The preference lattice

Gregorio Curello
University of Bonn

Ludvig Sinander
University of Oxford

2 April 2022

paper: arXiv.org/abs/1902.07260

Preference comparisons

Preference comparisons are ubiquitous:

- choice under risk/uncertainty:
  $\succeq'$ is *more risk-/ambiguity-averse* than $\succeq$
- monotone comparative statics:
  $\succeq'$ *takes larger actions* than $\succeq$
- dynamic problems:
  $\succeq'$ is *more delay-averse/impatient* than $\succeq$

All special cases of *single-crossing dominance.*

Outline

Study the *lattice structure* of single-crossing dominance:

characterisation, *existence* and *uniqueness* results for minimum upper bounds of arbitrary sets of preferences.

Applications:

- monotone comparative statics
- choice under risk/uncertainty
- social choice

Environment

Abstract environment is $(\mathcal{X}, \gtrsim)$:

– non-empty set $\mathcal{X}$ of alternatives...

– equipped with partial order $\gtrsim$.

Notation: $\mathcal{P}$ denotes set of all preferences on $\mathcal{X}$.

Single-crossing dominance S: for preferences $\succeq, \succeq' \in \mathcal{P}$,
$\succeq' \; S \succeq$ iff for any pair $x \gtrsim y$ of alternatives,
$x \succeq (\succ) \; y$ implies $x \succeq' (\succ') \; y$.

(Note: definition of S depends on $\gtrsim$.)

(Minimum) upper bounds

Let $P \subseteq \mathcal{P}$ be a set of preferences.

$\succeq' \in \mathcal{P}$ is an *upper bound* of P iff $\succeq' \, S \succeq$ for every $\succeq \in P$.

If also $\succeq'' \, S \succeq'$ for every (other) upper bound $\succeq''$ of P,
then $\succeq'$ is a *minimum* upper bound.

(MUB = 'join' = 'supremum')

Lattice structure

Study the *lattice structure* of $(\mathcal{P}, S)$:

(1) **characterisation theorem:**
characterisation of the minimum upper bounds of any set $P \subseteq \mathcal{P}$ of preferences.

(2) **existence theorem:**
necessary and sufficient condition on $\gtrsim$ for every set $P \subseteq \mathcal{P}$ to possess ≥ 1 minimum upper bound.
(The condition: $\gtrsim$ contains no *crowns* or *diamonds*.)

(3) **uniqueness proposition** (not today)**:**
necessary and sufficient condition on $\gtrsim$ for every set $P \subseteq \mathcal{P}$ to possess $= 1$ minimum upper bound.
(The condition: $\gtrsim$ is complete.)

Applications

Monotone comparative statics:

- group with preferences P
- consensus $C(P)$: alternatives optimal for every $\succeq \in P$
- comparative statics: when P increases, $C(P)$ increases.

Choice under uncertainty:

- study generalised maxmin preferences:
  those represented by $X \mapsto \inf_{\succeq \in P} c(\succeq, X)$ for some $P \subseteq \mathcal{P}$.
- characterisation: $\succeq^\star$ admits maxmin representation P
  iff $\succeq^\star$ a MUB of P w.r.t. 'more ambiguity-averse than'

Social choice:

- Sen's impossibility: $\{\text{strongly liberal}\} \cap \{\text{Pareto}\} = \varnothing$
- (im)possibility: n&s condition for $\{\text{liberal}\} \cap \{\text{Pareto}\} \neq \varnothing$

Plan

Characterisation theorem

Existence theorem

Application to monotone comparative statics

Application to ambiguity-aversion

P-chains

For alternatives $x \gtrsim y$,   a *P-chain from x to y* is a finite sequence $(w_k)_{k=1}^K$ such that

(1) $w_1 = x$ and $w_K = y$

(2) $w_k \gtrsim w_{k+1}, \quad \forall k < K$

(3) $w_k \succeq w_{k+1} \quad$ for some $\succeq \in P, \quad \forall k < K.$

Strict P-chain: $\quad w_k \succ w_{k+1} \quad$ for some $\succeq \in P, \quad \exists k < K.$

Example: $\quad \mathcal{X} = \{x, y, z\}, \quad x > y > z.$
$P = \{\succeq_1, \succeq_2\}, \quad$ where $\quad z \succ_1 x \succ_1 y \quad$ and $\quad y \succ_2 z \succ_2 x.$

P-chains, all strict: $\quad (x, y), \quad (y, z), \quad (x, y, z).$

Note: $\quad (x, z)$ is not a P-chain.

Characterisation theorem

Characterisation theorem.
For a preference $\succeq^\star \in \mathcal{P}$ and a set $P \subseteq \mathcal{P}$, TFAE:

(1) $\succeq^\star$ is a minimum upper bound of P.

(2) $\succeq^\star$ satisfies: for any $\gtrsim$-comparable $x, y \in \mathcal{X}$, wlog $x \gtrsim y$,

$(\star)$ $x \succeq^\star y$ iff $\exists$ P-chain from x to y, and

$(\star\star)$ $y \succeq^\star x$ iff $\nexists$ strict P-chain from x to y.

(Partial) proof of (2) implies (1)

Characterisation theorem.
For a preference $\succeq' \in \mathcal{P}$ and a set $P \subseteq \mathcal{P}$, TFAE:

(1) $\succeq^\star$ is a minimum upper bound of P.

(2) $\succeq^\star$ satisfies: for any $\gtrsim$-comparable $x, y \in \mathcal{X}$, wlog $x \gtrsim y$,

$$\begin{aligned} (\star)\quad & x \succeq^\star y \quad \text{iff} \quad \exists \phantom{\text{ strict}}\ P\text{-chain from } x \text{ to } y, \quad \text{and} \\ (\star\star)\quad & y \succeq^\star x \quad \text{iff} \quad \nexists \text{ strict } P\text{-chain from } x \text{ to } y. \end{aligned}$$

---

(2) $\Longrightarrow$ (1), upper bound: WTS $\succeq^\star S \succeq$ for every $\succeq \in P$:

$$x \gtrsim y \quad \text{and} \quad x \succeq y \quad \Longrightarrow \quad x \succeq^\star y.$$

Holds by $(\star)$ because (x, y) is a P-chain.

(Partial) proof of (2) implies (1)

Characterisation theorem.
For a preference $\succeq' \in \mathcal{P}$ and a set $P \subseteq \mathcal{P}$, TFAE:

(1) $\succeq^\star$ is a minimum upper bound of P.

(2) $\succeq^\star$ satisfies: for any $\gtrsim$-comparable $x, y \in \mathcal{X}$, wlog $x \gtrsim y$,

$$(\star)\ x \succeq^\star y \quad \text{iff} \quad \exists \quad P\text{-chain from } x \text{ to } y, \quad \text{and}$$
$$(\star\star)\ y \succeq^\star x \quad \text{iff} \quad \nexists \quad \text{strict } P\text{-chain from } x \text{ to } y.$$

---

(2) $\Longrightarrow$ (1), minimum: WTS $\succeq' S \succeq^\star$ for every UB $\succeq'$ of P:

$$x \gtrsim y \quad \text{and} \quad x \succeq^\star y \quad \Longrightarrow \quad x \succeq' y.$$

By $(\star)$, $\exists$ P-chain $(w_k)_{k=1}^K$ from x to y:
$\forall k < K$, $w_k \gtrsim w_{k+1}$ and $w_k \succeq w_{k+1}$ for some $\succeq \in P$

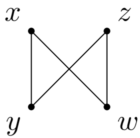
$\Longrightarrow$ $w_k \succeq' w_{k+1}$ because $\succeq'$ is an UB of P

$\Longrightarrow$ $x \succeq' y$ since $\succeq' \in \mathcal{P}$ is transitive.

Plan

Failure of existence

Example: $\mathcal{X} = \{x, y, z, w\}$ with following partial order $\gtrsim$:



$P = \{\succeq_1, \succeq_2\} \subseteq \mathcal{P}$, where

$$w \succ_1 x \succ_1 y \succ_1 z \qquad \text{and} \qquad y \succ_2 z \succ_2 w \succ_2 x.$$

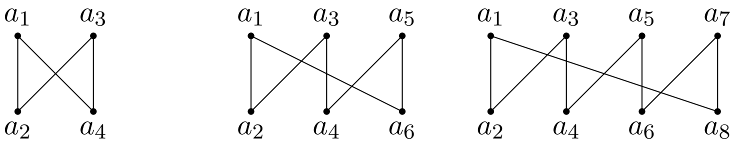
$\exists$ strict P-chain $x \to y$ and $z \to w$ $\implies$ $x \succ^\star y$ and $z \succ^\star w$

$\nexists$ P-chain $x \to w$ or $z \to y$ $\implies$ $w \succ^\star x$ and $y \succ^\star z$

Then $x \succ^\star y \succ^\star z \succ^\star w \succ^\star x$. Not a preference! ($\notin \mathcal{P}$)

Crowns

Same idea applies whenever $\succsim$ contains a *crown:*



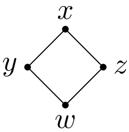
(a) A 4-crown. (b) A 6-crown. (c) An 8-crown.

A K-*crown* ($K \geq 4$ even) is a sequence $(a_k)_{k=1}^{K}$ in $\mathcal{X}$ s.t.

- $a_{k-1} > a_k < a_{k+1} \quad$ for $1 < k \leq K$ even ($a_{K+1} \equiv a_1$)
- non-adjacent $a_k, a_{k'}$ are $\succsim$-incomparable.

Diamonds

Existence also fails when $\gtrsim$ contains a *diamond:*



A *diamond* is (x, y, z, w) such that $x > y > w$ and $x > z > w$, but y, z are incomparable.

(existence failure example on slide 34)

Existence theorem

But that's all:

Existence theorem. The following are equivalent:

(1) Every set of preferences has ≥ 1 minimum upper bound.

(2) $\succsim$ is crown- and diamond-free.

Special cases:

- (2) holds whenever there are ≤ 3 alternatives
- (2) holds if $\succsim$ is complete
- (2) fails for any lattice that isn't a chain (=totally ordered)

Proof $\neg(2) \implies \neg(1)$: by counter-example.

Proof $(2) \implies (1)$: non-trivial.
(Relies on Suzumura's extension theorem.)

Plan

Characterisation theorem

Existence theorem

Application to monotone comparative statics

Application to ambiguity-aversion

Monotone comparative statics

Let $\mathcal{X} \subseteq \mathbf{R}$ be a set of actions, ordered by inequality $\geq$.

Argmax of a preference $\succeq \in \mathcal{P}$:

$$X(\succeq) \coloneqq \{x \in \mathcal{X} : x \succeq y \text{ for every } y \in \mathcal{X}\}.$$

Consensus among a group with preferences $P \subseteq \mathcal{P}$:

$$C(P) \coloneqq \bigcap_{\succeq \in P} X(\succeq).$$

Comparative statics question:
what shifts of P cause consensus $C(P)$ to 'increase'?

Standard theory

For $X, X' \subseteq \mathcal{X}$,
X' dominates X in the *($\geq$-induced) strong set order* iff
for any $x \in X$ and $x' \in X'$,
the meet (join) of $\{x, x'\}$ lies in X (in X').

Theorem.[1] For $\succeq, \succeq' \in \mathcal{P}$, if $\succeq' \; S \succeq$,
then $X(\succeq')$ dominates $X(\succeq)$ in the ($\geq$-induced) strong set order.

---

[1] Milgrom and Shannon (1994) and LiCalzi and Veinott (1992).

Consensus comparative statics

$\geq$ is complete $\implies$ crown- and diamond-free
$\implies$ every set of preferences has ≥ 1 meet and join.

For $P, P' \subseteq \mathcal{P}$,
P' dominates P in the *(S-induced) strong set order* iff
for any $\succeq \in P$ and $\succeq' \in P'$,
the meet (join) of $\{\succeq, \succeq'\}$ lies in P (in P').

Proposition. For $P, P' \subseteq \mathcal{P}$,
if P' dominates P in the (S-induced) strong set order,
then $C(P')$ dominates $C(P)$ in the ($\geq$-induced) strong set order.

Proof

Take $x \in C(P)$ and $x' \in C(P')$;
Must show $x \wedge x' \in C(P)$ and $x \vee x' \in C(P')$.

Take arbitrary $\succeq \in P$ and $\succeq' \in P'$. Note $x \in C(P) \subseteq X(\succeq)$.

By existence theorem, $\exists$ minimum upper bound $\succeq^\star$ of $\{\succeq, \succeq'\}$.

Since P' dominates P in the SSO, $\succeq^\star$ lies in P'.
$\implies x' \in C(P') \subseteq X(\succeq^\star)$.

Since $\succeq^\star S \succeq$, $X(\succeq^\star)$ dominates $X(\succeq)$ in the SSO
by the standard theorem two slides back.

$\implies x \wedge x' \in X(\succeq)$.

Since $\succeq \in P$ was arbitrary,
$\implies x \wedge x' \in \bigcap_{\succeq \in P} X(\succeq) = C(P)$.

Plan

Characterisation theorem

Existence theorem

Application to monotone comparative statics

Application to ambiguity-aversion

Choice under uncertainty

Standard Savage framework:

- states of the world Ω
- monetary prizes $\Pi \subseteq \mathbf{R}$
- a set $\mathcal{X}$ of *acts*, meaning functions $X : \Omega \to \Pi$
- the subset of *constant acts* is denoted $\mathcal{C} \subseteq \mathcal{X}$

Notation: $\mathcal{P}$ is the set of all preferences (no axioms) on $\mathcal{X}$.

Definition.[2] For preferences $\succeq, \succeq' \in \mathcal{P}$ over acts,
$\succeq'$ *is more ambiguity-averse than* $\succeq$
iff for any act $X \in \mathcal{X}$ and constant act $C \in \mathcal{C}$,
$C \succeq(\succ)\, X \quad \Longrightarrow \quad C \succeq'(\succ')\, X$.

[2] Ghirardato and Marinacci (2002) and Epstein (1999).

‘More ambiguity-averse than’ as single-crossing

Definition. For preferences $\succeq, \succeq' \in \mathcal{P}$ over acts,
$\succeq'$ *is more ambiguity-averse than* $\succeq$,
iff for any act $X \in \mathcal{X}$ and constant act $C \in \mathcal{C}$,
$C \succeq (\succ)\, X \quad \Longrightarrow \quad C \succeq' (\succ')\, X$.

Define $\gtrsim$ on $\mathcal{X}$ as follows:
for acts $X, Y \in \mathcal{X}$, $\quad X \gtrsim Y$ iff either

(i) X is constant and Y is not, or

(ii) $X = Y$.

‘More ambiguity-averse than’ is precisely
single-crossing dominance S as induced by $\gtrsim$.

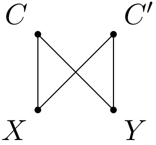
Choice under uncertainty: failure of existence

'More ambiguity-averse than' is S as induced by $\succsim$, where $X \succsim Y$ iff either

(i) X is constant and Y is not, or

(ii) $X = Y$.

$\succsim$ contains crowns!



$\implies$ not all sets of preferences possess minimum upper bounds.

Existence

Let's restrict attention to monotone preferences:

Preference $\succeq \in \mathcal{P}$ is *monotone*
iff for any constant acts $C, D \in \mathcal{C}$, $\quad C \succeq D \quad$ iff $\quad C \geq D$.

Augment the definition of $\gtrsim$: $\quad X \gtrsim' Y$ iff either
(i) X is constant and Y is not,
(ii) $X = Y$, or
(iii) X, Y are constant and $X \geq Y$.

All monotone preferences agree with $\gtrsim'$ on pairs of type (iii).
$\Longrightarrow$ for monotone preferences, 'more ambiguity-averse than' coincides with S as induced by $\gtrsim'$.

And $\gtrsim'$ is crown- and diamond-free.
$\Longrightarrow$ every set of *monotone* preferences has
≥ 1 minimum upper bound w.r.t. 'more ambiguity-averse than'.

Solvability

A *certainty equivalent* for $\succeq \in \mathcal{P}$ of an act $X \in \mathcal{X}$
is a prize $c(\succeq, X) \in \Pi$ such that $X \succeq c(\succeq, X) \succeq X$.

A preference with a certainty equivalent for every act is called *solvable.*

Maxmin representations

Definition. A set $P \subseteq \mathcal{P}$ of monotone and solvable preferences is a *maxmin representation* of a preference $\succeq^\star \in \mathcal{P}$ iff

$$X \mapsto \inf_{\succeq \in P} c(\succeq, X)$$

ordinally represents $\succeq^\star$.

Maxmin expected utility[3] is a special case:
P a set of *expected-utility* preferences with
the same (strictly increasing) u but different beliefs $\mu_{\succeq}$.

$$\begin{aligned} X \mapsto \inf_{\succeq \in P} c(\succeq, X) &= \inf_{\succeq \in P} u^{-1}\left(\int_\Omega [u \circ X] \mathrm{d}\mu_{\succeq}\right) \\ &= u^{-1}\left(\inf_{\succeq \in P} \int_\Omega [u \circ X] \mathrm{d}\mu_{\succeq}\right) \end{aligned}$$

[3] Gilboa and Schmeidler (1989).

Maxmin–join equivalence

Proposition. For a preference $\succeq^\star \in \mathcal{P}$ and a set $P \subseteq \mathcal{P}$ of monotone and solvable preferences over acts, TFAE:

(1) P is a maxmin representation of $\succeq^\star$.

(2) $\succeq^\star$ is a minimum upper bound of P w.r.t. 'more ambiguity-averse than'.

Proof relies on the characterisation theorem. (slide 35)

(Trivial) representation theorem

Entire maxmin class is too broad to restrict behaviour much:

Proposition. A preference over acts admits a maxmin representation iff it is monotone and solvable.

$\Longleftarrow$: if $\succeq^\star$ is monotone & solvable then $\{\succeq^\star\}$ is a maxmin representation.

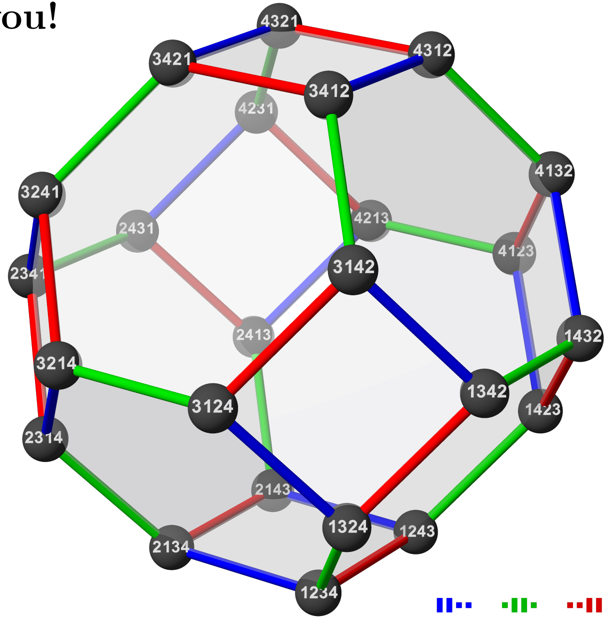
$\Longrightarrow$: suppose $\succeq^\star$ admits maxmin representation P.

Solvable: certainty equivalent of X is $\inf_{\succeq \in P} c(\succeq, X)$.

Monotone: on the constant acts $\mathcal{C}$, $\succeq^\star$ is represented by

$$C \mapsto \inf_{\succeq \in P} \underbrace{c(\succeq, C)}_{=C} = C.$$

Thank you!

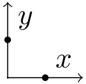


The lattice of strict preferences over $\mathcal{X} = \{1, 2, 3, 4\}$.

Failure of uniqueness

Consider

$$\mathcal{X} = \{\underbrace{(1,0)}_{x}, \underbrace{(0,1)}_{y}\} \quad =$$



Observe: $\succeq' \; S \; \succeq$ holds for *any* $\succeq, \succeq' \in \mathcal{P}$:

'for any $\gtrsim$-comparable pair of alternatives $x, y \in \mathcal{X}$, wlog $x \gtrsim y$,
$x \succeq (\succ) \; y \quad \implies \quad x \succeq' (\succ') \; y$.'

Holds vacuously (*no* pairs are $\gtrsim$-comparable).

$\implies$ every $\succeq \in \mathcal{P}$ is a minimum upper bound of every $P \subseteq \mathcal{P}$.

Uniqueness

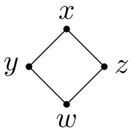
Uniqueness proposition. The following are equivalent:

(1) Every set of preferences has ≤ 1 minimum upper bound.

(2) Every set of preferences has $= 1$ minimum upper bound.

(3) $\gtrsim$ is complete.

Failure of existence for diamonds



Existence fails for $P = \{\succeq_1, \succeq_2\} \subseteq \mathcal{P}$, where

$$y \succ_1 w \succ_1 z \succ_1 x \qquad \text{and} \qquad w \succ_2 z \succ_2 x \succ_2 y.$$

$\exists$ strict P-chain $x \to w$ (viz. (x, y, w))
$\nexists$ P-chain $z \to w$ or $x \to z$

$\implies$ $x \succ^\star w \succ^\star z \succ^\star x$. Not a preference! ($\notin \mathcal{P}$)

(back to slide 15)

Proof of maxmin–join equivalence

By characterisation theorem, suffices to show that for $X \gtrsim' Y$, $\exists$ (strict) P-chain $X \to Y$ iff

$$\inf_{\succeq \in P} c(\succeq, X) \geq (>) \inf_{\succeq \in P} c(\succeq, Y).$$

$X = C$ constant, Y not: the following are equivalent:

- $\exists$ (strict) P-chain from C to Y.
- $C \succeq' (\succ') \, Y$ for some preference $\succeq' \in P$.
- $\inf_{\succeq \in P} c(\succeq, C) \geq (>) \inf_{\succeq \in P} c(\succeq, Y)$.

$X = C$, $Y = D$ both constant: the following are equivalent:

- $\exists$ (strict) P-chain from C to D.
- $C \geq (>) \, D$.
- $\inf_{\succeq \in P} c(\succeq, C) \geq (>) \inf_{\succeq \in P} c(\succeq, D)$.

(back to slide 29)

References

Epstein, L. G. (1999). A definition of uncertainty aversion. *Review of Economic Studies*, *6*(3), 579–608. https://doi.org/10.1111/1467-937X.00099

Ghirardato, P., & Marinacci, M. (2002). Ambiguity made precise: A comparative foundation. *Journal of Economic Theory*, *102*(2), 251–289. https://doi.org/10.1006/jeth.2001.2815

Gilboa, I., & Schmeidler, D. (1989). Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, *18*(2), 141–153. https://doi.org/10.1016/0304-4068(89)90018-9

LiCalzi, M., & Veinott, A. F. (1992). *Subextremal functions and lattice programming* [working paper]. https://doi.org/10.2139/ssrn.877266

Milgrom, P., & Shannon, C. (1994). Monotone comparative statics. *Econometrica*, *62*(1), 157–180. https://doi.org/10.2307/2951479