Agenda-manipulation in ranking

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Ranking by committee

A committee must rank a set of alternatives.

Hiring:

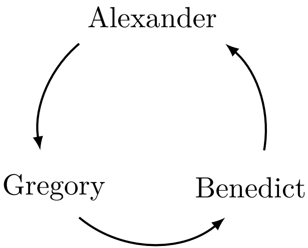
- alternatives are candidates for a job
- uncertainty about who will accept
- hiring committee decides to whom to offer the job, to whom next if the first candidate declined, etc.

Party lists:

- alternatives are a political party's parliamentary candidates
- party's leadership committee ranks them ('party list')
- the K highest-ranked candidates get parliamentary seats, where K is (uncertain) # seats the party wins in an election

Interaction

The majority will may contain (Condorcet) cycles:



The committee's *chair* chooses the order of pairwise votes.

Transitivity is imposed.

Preferences

The chair has a preference $\succ$ over alternatives.

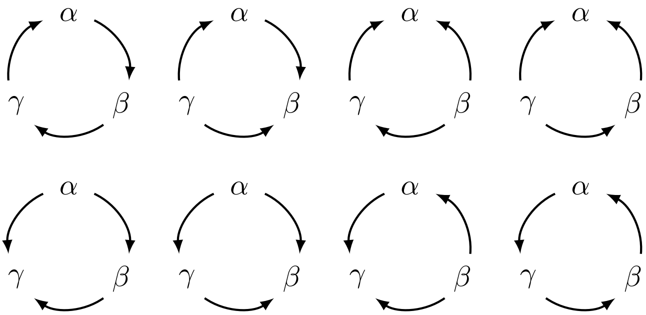
Ranking R is *more aligned with* $\succ$ than R'
iff whenever $x \succ y$ and $x \; R' \; y$, also $x \; R \; y$.

The chair prefers rankings that are more aligned with $\succ$.

Hiring: a more aligned ranking is exactly one that hires a $\succ$-better candidate at every realisation of uncertainty.

Unknown majority will

The chair does not know the majority will, W.



Regret-free strategies

A ranking is W*-unimprovable* iff no other ranking is both

(i) reachable under W and

(ii) more aligned with $\succ$.

With perfect knowledge of W,
W-unimprovability is the strongest optimality concept.

A *regret-free* strategy
reaches a W-unimprovable ranking under every W.

Results

We introduce a strategy called *insertion sort.*

Theorem 1.
Insertion sort is regret-free.

What (other) strategies are regret-free?
Theorem 2: characterisation of *outcomes.*
Theorem 3: characterisation of *behaviour.*

What's special about insertion sort?
Theorem 4: IS is characterised by a lexicographic property.

Related literature

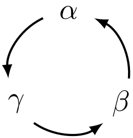
- **agenda-manipulation:** Black (1958), Farquharson (1969), Miller (1977), Banks (1985)

  … with incomplete info: Ordeshook and Palfrey (1988), recent work by Benny Moldovanu & co-authors

- **social choice:** Zermelo (1929), Wei (1952), Kendall (1955)
  - Copeland's method: Copeland (1951), Rubinstein (1980)
  - Kemeny–Slater method: Kemeny (1959), Slater (1961), Young and Levenglick (1978), Young (1986, 1988)
  - fair-bets method: Daniels (1969), Moon and Pullman (1970), Slutzki and Volij (2005)

(references: slide 29)

Example



and $\alpha \succ \beta \succ \gamma$

Rankings reachable under W:
$\beta \; R \; \alpha \; R \; \gamma$, $\alpha \; R' \; \gamma \; R' \; \beta$ and $\gamma \; R'' \; \beta \; R'' \; \alpha$.

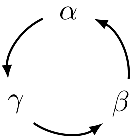
R and R' are more aligned with $\succ$ than R''
and are incomparable to each other.

$\Longrightarrow$ R and R' are W-unimprovable.

Efficiency

A W-*efficient* ranking
is one that ranks x above y whenever both $x \succ y$ and $x\ W\ y$.

Example.



and $\quad \alpha \succ \beta \succ \gamma$

W-efficient rankings: $\quad \succ$ itself, $\quad \beta\ R\ \alpha\ R\ \gamma \quad$ and $\quad \alpha\ R'\ \gamma\ R'\ \beta$.

Definition.
A strategy is *efficient* iff for any majority will W,
its outcome under W is W-efficient.

W-efficiency implies W-unimprovability

Lemma 1.
For any majority will W,
a W-efficient ranking is W-unimprovable.

Corollary.
Any efficient strategy is regret-free.

Proof of Lemma 1

Fix a W, a W-efficient R, and a W-reachable $R' \neq R$.
Suppose toward a contradiction that R' is MAW $\succ$ than R.

Since $R' \neq R$, $\exists$ alternatives x, y such that $x\ R'\ y$ and $y\ R\ x$.
Enumerate the alternatives that R' ranks between x and y as

$$x = z_1\ R'\ z_2\ R' \cdots R'\ z_N = y.$$

Since R' is W-reachable, we must have $z_1\ W\ z_2\ W \cdots W\ z_N$.

There has to be $n < N$ at which $z_{n+1}\ R\ z_n$,
else we'd have $x\ R\ y$ by transitivity of R.

It must be that $z_{n+1} \succ z_n$,
else we'd have $z_n\ R\ z_{n+1}$ by $z_n\ W\ z_{n+1}$ and W-efficiency of R.

So (z_n, z_{n+1}) is ranked 'right' by R and 'wrong' by R'
... which is absurd since R' is MAW $\succ$ than R. ∎

Insertion sort

Label the alternatives $\{1, \ldots, n\}$ so that $1 \succ \cdots \succ n$.

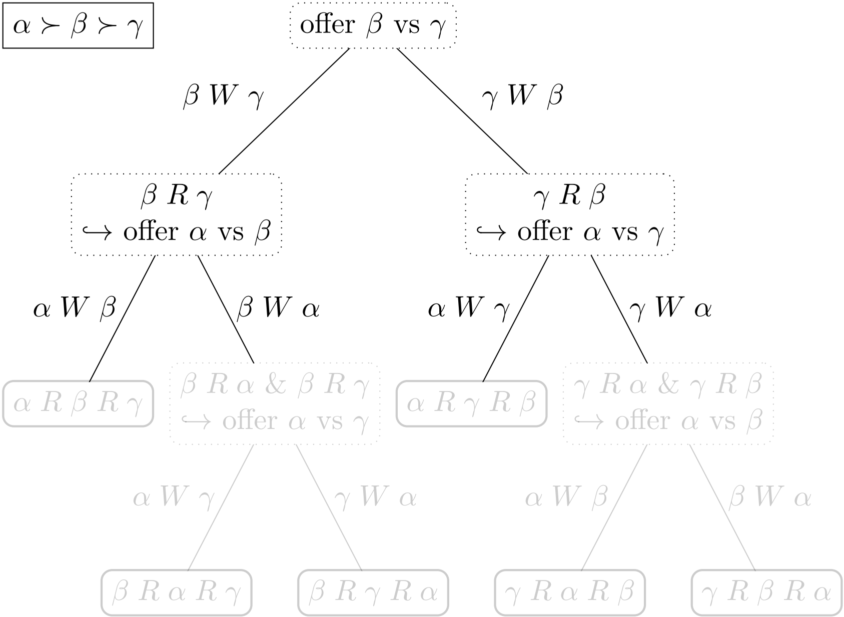
Insertion sort strategy: for each $k \in \{n-1, \ldots, 1\}$,

– totally rank $\{k+1, \ldots, n\}$

(write $x_{k+1} \; R \cdots R \; x_n$, where $\{x_{k+1}, \ldots, x_n\} \equiv \{k+1, \ldots, n\}$)

– 'insert' k into $\{k+1, \ldots, n\}$:

pit k against the highest-ranked (x_{k+1});
then (if k lost) pit k against the 2^{nd}-highest-ranked (x_{k+2});
…


$\alpha \succ \beta \succ \gamma$
offer β vs γ
$\beta\, W\, \gamma$
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$\beta\, R\, \gamma$ $\hookrightarrow$ offer α vs β
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Insertion sort is regret-free

Theorem 1.
The insertion-sort strategy is efficient, hence regret-free.

Proof of Theorem 1

Fix a W, and let R be the outcome of IS under W.
Fix x, y with $x \succ y$ and $x \; W \; y$; we must show that $x \; R \; y$.

Enumerate all alternatives $\succ$-worse than x as $\quad z_1 \; R \cdots R \; z_K$.
Note that $z_k = y$ for some $k \leq K$.

By definition of IS,
x is pitted against $z_1, z_2, \ldots$ in turn until it wins a vote.

- if x loses against $z_1, \ldots, z_{k-1}$,
  then it is pitted against $z_k = y$ and wins (since $x \; W \; y$)
  $\implies x \; R \; y$.

- if x wins against z_ℓ for $\ell < k$,
  then $x \; R \; z_\ell \; R \cdots R \; z_k = y$
  $\implies x \; R \; y$ (by transitivity of R). ■

What (other) strategies are regret-free?

We've shown that regret-free strategies exist.

What are their characteristics?

Characterisation of outcomes

Recall that W-efficiency $\Longrightarrow$ W-unimprovability (Lemma 1).

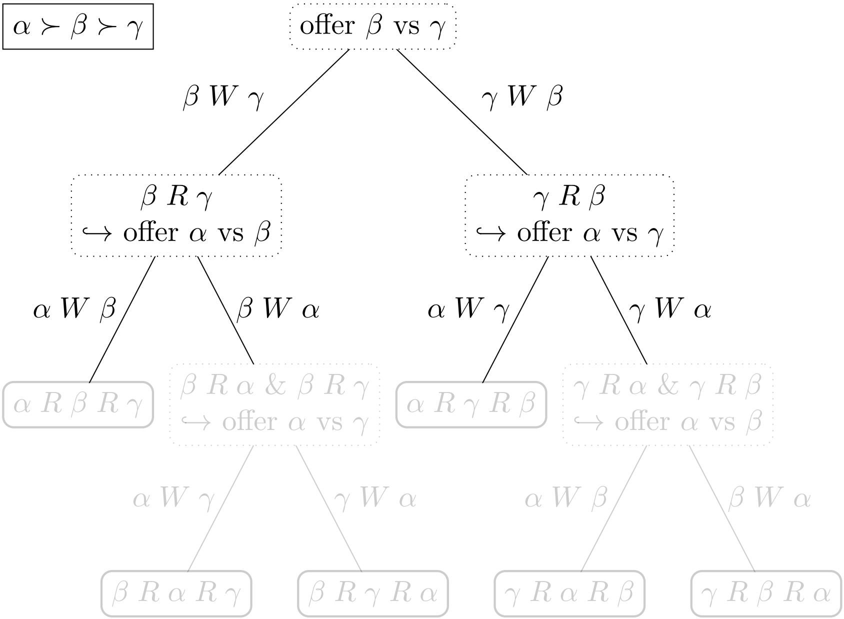
The converse is false:
a W-unimprovable ranking need not be W-efficient.

(counter-example: slide 24)

But only efficiency ensures unimprovability robustly across Ws:

Theorem 2.
A strategy is regret-free iff it is efficient.

(tightness: slide 25)


α ≻ β ≻ γ
offer β vs γ
β W γ
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α W γ
γ W α
α W β
β W α
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β R γ R α
γ R α R β
γ R β R α

Characterisation of behaviour

Theorem 3.
A strategy is regret-free iff
it never misses an opportunity or takes a risk.

(formal definitions: slide 26) (tightness: slide 27)

History-invariant voting

We have assumed throughout that W is fixed
$\iff$ voting is (approximately) history-invariant.

Reasonable if voters are unsophisticated or vote expressively.

Not unreasonable if voting is strategic. (details: slide 28)

Alexander der fünfft

THANKS!

Gregorius der.xij.

Benedictus der.xiij.

(anti-)popes in 1409–10, from Schedel (1493)

What's special about insertion sort?

For an alternative x, strategy σ and majority will W,
write $R^\sigma(W)$ for the outcome of σ under W, and

$$N_x^\sigma(W) \coloneqq |\{y : x \succ y \text{ and } x\ R^\sigma(W)\ y\}|.$$

Definition.
Given an alternative x, σ is *better for x than* σ' iff

$$|\{W : N_x^\sigma(W) \geq k\}| \geq |\{W : N_x^{\sigma'}(W) \geq k\}| \quad \forall k \in \{1, \ldots, n-1\}.$$

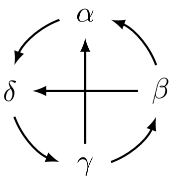
If $\sigma \in \Sigma$ is better for x than each $\in \Sigma$, it is *best for x among Σ.*

Label the alternatives $\{1, \ldots, n\}$ so that $1 \succ \cdots \succ n$.

Theorem 4.
A strategy is outcome-equivalent to insertion sort iff
among all strategies, it is best for 1;
among such strategies, it is best for 2; and so on.

Counter-example to the converse of Lemma 1

Alternatives $\{\alpha, \beta, \gamma, \delta\}$ with $\alpha \succ \beta \succ \gamma \succ \delta$ and



The ranking $\alpha \; R \; \delta \; R \; \gamma \; R \; \beta \ldots$

(– is reachable under W: offer $\{\alpha, \delta\}$, $\{\delta, \gamma\}$, $\{\gamma, \beta\}$.)

– is W-unimprovable,
since no other W-reachable ranking ranks α above β.
(Because there's only one directed path in W from α to β.)

– is not W-efficient, since $\delta \; R \; \beta$.

(back to slide 18)

Theorem 2 tightness

The characterisation in Therem 2 is tight in the following sense:

Proposition 1.
For any W and W-reachable W-efficient ranking R, some regret-free strategy has outcome R under W.

Thus for every majority will W,

$$\begin{aligned} & \{R : \exists \text{ regret-free strategy with outcome } R \text{ under } W\} \\ = & \{R : R \text{ is } W\text{-reachable and } W\text{-efficient}\} \end{aligned}$$

($\subseteq$ by Theorem 2, $\supseteq$ by Proposition 1)

(back to slide 18)

Formal definition of errors

A *proto-ranking* is an incomplete ranking: formally,
an irreflexive and transitive relation on the set of alternatives.

Definition.
Let R be a non-total proto-ranking, and let $x \succ y$ be unranked.

(1) Offering $\{x, y\}$ for a vote *misses an opportunity (at R)* iff there is an alternative z s.t. $x \succ z \succ y$ and $y \not R\, z \not R\, x$.

(2) Offering $\{x, y\}$ for a vote *takes a risk (at R)* iff there is an alternative z s.t. either

- $z \succ y$, $x\, R\, z$ and $y \not R\, z$, or
- $x \succ z$, $z\, R\, y$ and $z \not R\, x$.

(back to slide 20)

Theorem 3 tightness

Proposition 2.
After any error-free history,
there is a pair that can be offered without committing an error.

Yields tightness:
for any W and any sequence of pairs that is error-free under W,
some regret-free strategy offers this sequence under W.

(back to slide 20)

Strategic voting

Each voter i has a preference $\succ_i$ over alternatives,
and prefers rankings more aligned with $\succ_i$.

A voter's *strategy* specifies how to vote at each history.
The *sincere strategy:* vote for your favourite. History-invariant!

Outcome of chair [voters] using σ [σ_i, σ_{-i}] denoted $R(\sigma, \sigma_i, \sigma_{-i})$.

Definition.
A strategy σ_i is *dominant* iff for any alternative strategy σ'_i,

($\nexists$) there exists no profile σ, σ_{-i} such that $R(\sigma', \sigma_i, \sigma_{-i})$ is distinct from, and MAW $\succ_i$ than, $R(\sigma, \sigma_i, \sigma_{-i})$.

($\exists$) there exists a profile σ, σ_{-i} such that $R(\sigma, \sigma_i, \sigma_{-i})$ is distinct from, and MAW $\succ_i$ than, $R(\sigma', \sigma_i, \sigma_{-i})$.

Proposition 4.

The sincere strategy is (uniquely) dominant.

(back to slide 21)

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