The converse envelope theorem

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Envelope theorem: optimal decision-making $\implies$ $\boxtimes$ formula.

Textbook intuition: $\boxtimes$ formula $\iff$ FOC.

Modern envelope theorem of MS02:* almost no assumptions.

$\hookrightarrow$ FOC ill-defined, so need different intuition.

My theorem: with almost no assumptions,
$\boxtimes$ formula equivalent to generalised FOC.

- an envelope theorem: FOC $\implies$ $\boxtimes$
- *a converse:* $\boxtimes$ $\implies$ FOC.

Application to mechanism design.

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*Milgrom, P., & Segal, I. (2002). Envelope theorems for arbitrary choice sets. *Econometrica*, *70*(2), 583–601. doi:10.1111/1468-0262.00296

Setting

Agent chooses action x from a set $\mathcal{X}$.
Objective $f(x,t)$, where $t \in [0,1]$ is a parameter.

No assumptions on $\mathcal{X}$, almost none on f:

(1) $f(x,\cdot)$ is differentiable for each $x \in \mathcal{X}$

(2) $f(x,\cdot)$ is 'not too erratic'.

Decision rule: a map $X : [0,1] \to \mathcal{X}$.

Associated *value function:* $V_X(t) \coloneqq f(X(t),t)$.

Envelope theorem

X *satisfies the* $\boxtimes$ *formula* iff

$$V_X(t) = V_X(0) + \int_0^t f_2(X(s), s)\mathrm{d}s \quad \text{for every } t \in [0,1].$$

Equivalently: V_X is absolutely continuous and

$$V_X'(t) = f_2(X(t), t) \quad \text{for a.e. } t \in (0,1).$$

X is *optimal* iff for every t, $X(t)$ maximises $f(\cdot, t)$.

Modern envelope theorem (MS02).
Any optimal decision rule satisfies the $\boxtimes$ formula.

Textbook intuition

Differentiation identity for $V_X(t) \coloneqq f(X(t),t)$:

$$V_X'(t) = \underbrace{\frac{\mathrm{d}}{\mathrm{d}m} f(X(t+m),t)\Big|_{m=0}}_{\text{`indirect effect'}} + \underbrace{f_2(X(t),t)}_{\text{`direct effect'}}.$$

$$\begin{aligned} & V_X'(t) = \text{direct effect} && (\boxtimes\text{ formula}) \\ \iff \quad & \text{indirect effect} = 0 && (\text{FOC}). \end{aligned}$$

Problem: ‘indirect effect’ (hence FOC) ill-defined!

– $f(\cdot,t)$ & X need not be differentiable.

– actions $\mathcal{X}$ need have no convex or topological structure.

The outer first-order condition

Disjuncture: in general, $\boxtimes$ formula $\not\Leftrightarrow$ FOC.

- one solution: add strong 'classical' assumptions.
- my solution: find the correct FOC!

Decision rule X *satisfies the outer FOC* iff

$$\frac{\mathrm{d}}{\mathrm{d}m} \int_r^t f(X(s+m), s)\mathrm{d}s \bigg|_{m=0} = 0 \quad \text{for all } r, t \in (0,1).$$

'Integrated' version of classical FOC.

- always well-defined & necessary for optimality
- equiv'nt to classical FOC when latter well-defined.

Theorem

Envelope theorem & converse.
For a decision rule $X : [0,1] \to \mathcal{X}$, the following are equivalent:

(1) X satisfies the oFOC,
and $V_X(t) \coloneqq f(X(t), t)$ is absolutely continuous.

(2) X satisfies the ⌧ formula

$$V_X(t) = V_X(0) + \int_0^t f_2(X(s), s)\mathrm{d}s \quad \text{for every } t \in [0,1].$$

Application: environment

Agent with preferences $f(y, p, t)$ over
physical outcome $y \in \mathcal{Y}$ and payment $p \in \mathbf{R}$.

- $\mathcal{Y}$ partially ordered
- type $t \in [0, 1]$ is agent's private info
- assume single-crossing.

A *physical allocation* is $Y : [0, 1] \to \mathcal{Y}$.

Y is *implementable* iff $\exists$ payment rule $P : [0, 1] \to \mathbf{R}$
s.t. (Y, P) is incentive-compatible.

Application: goal

Classical result: implementable $\iff$ increasing.

'$\Longleftarrow$' is the substantial part. Versions:

	literature	this paper
outcomes $\mathcal{Y}$	$\subseteq \mathbf{R}$	general
preferences f	quasi-linear	general.

Application: result

Implementability theorem. Under regularity assumptions, any increasing physical allocation is implementable.

Argument:

- fix an increasing physical allocation $Y : [0,1] \to \mathcal{Y}$
- choose a payment rule P so that $\boxtimes$ formula holds
- then by *converse envelope theorem*, oFOC holds $\iff$ mechanism (Y, P) is locally IC.
- finally, local IC $\implies$ global IC by single-crossing.

Application: example

Monopolist selling information.

Physical allocations $\mathcal{Y}$:
distributions of posterior beliefs, ordered by Blackwell.

By the implementability theorem, any information allocation that gives higher types Blackwell-better signals can be implemented.

Plan

Setting and background
- Setting
- Classical envelope theorem and converse

Main theorem
- The outer first-order condition
- Envelope theorem and converse

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- Environment and result
- Selling information

Refresher: absolute continuity

$$\text{Lipschitz} \implies \text{AC} \implies \begin{array}{c}\text{uniformly continuous}\\ \text{\& diff'able a.e.}\end{array}$$

Lebesgue's fundamental theorem of calculus.
For $\phi : [0,1] \to \mathbf{R}$, the following are equivalent:

(1) ϕ is AC.

(2) ϕ is diff'able a.e. & $\phi(t) = \phi(0) + \int_0^t \phi' \quad \forall\, t \in [0,1]$.

Refresher: uniform integrability

Family $\{\phi_i\}_{i \in I}$ of functions $[0,1] \to \mathbf{R}$ is *uniformly integrable*

iff for any $\varepsilon > 0, \quad \exists\, \delta > 0 \quad$ s.t.

$$\sup_{i \in I} \int_T |\phi_i| < \varepsilon \quad \text{for any } T \subseteq [0,1] \text{ of measure } < \delta.$$

Vitali convergence theorem.
If a sequence $\{\phi_n\}_{n \in \mathbf{N}}$ of functions $[0,1] \to \mathbf{R}$
is UI & a.e. convergent, then $\lim_{n \to \infty} \phi_n$ is integrable and

$$\lim_{n \to \infty} \int_r^t \phi_n = \int_r^t \lim_{n \to \infty} \phi_n \quad \text{for any } r, t.$$

bounded $\implies$ dominated by integrable f'n $\implies$ UI.

Setting

Agent chooses action x from a set $\mathcal{X}$.
Objective $f(x,t)$, where $t \in [0,1]$ is a parameter.

No assumptions on $\mathcal{X}$, almost none on f:

(1) $f(x,\cdot)$ is differentiable for each $x \in \mathcal{X}$

(2) $f(x,\cdot)$ is 'not too erratic'.

Absolute equi-continuity

A family $\{\phi_x\}_{x\in\mathcal{X}}$ of functions $[0,1] \to \mathbf{R}$ is *absolutely equi-continuous (AEC)* iff the family

$$\left\{ t \mapsto \sup_{x\in\mathcal{X}} \left| \frac{\phi_x(t+m) - \phi_x(t)}{m} \right| \right\}_{m>0}$$

is uniformly integrable. (properties: slide 45)

'$f(x,\cdot)$ not too erratic': $\{f(x,\cdot)\}_{x\in\mathcal{X}}$ is AEC.

- a sufficient condition (maintained by MS02):
  - $f(x,\cdot)$ absolutely continuous for each $x \in \mathcal{X}$, and
  - $t \mapsto \sup_{x\in\mathcal{X}} |f_2(x,t)|$ dominated by an integrable f'n.
- a stronger sufficient condition: f_2 bounded.

Setting

Agent chooses action x from a set $\mathcal{X}$.
Objective $f(x,t)$, where $t \in [0,1]$ is a parameter.

No assumptions on $\mathcal{X}$, almost none on f:

(1) $f(x,\cdot)$ is differentiable for each $x \in \mathcal{X}$

(2) $f(x,\cdot)$ is 'not too erratic'.

Decision rule: a map $X : [0,1] \to \mathcal{X}$.

Associated *value function:* $V_X(t) \coloneqq f(X(t),t)$.

Envelope theorem

X *satisfies the* ⌧ *formula* iff

$$V_X(t) = V_X(0) + \int_0^t f_2(X(s), s)\mathrm{d}s \quad \text{for every } t \in [0, 1].$$

X is *optimal* iff for every t, $X(t)$ maximises $f(\cdot, t)$.

Modern envelope theorem (MS02).[†]
Any optimal decision rule satisfies the ⌧ formula.

[†] Really a slight refinement of MS02.

Plan

Classical assumptions

Classical assumptions:

- $\mathcal{X}$ is a convex subset of $\mathbf{R}^n$
- action derivative f_1 exists & is bounded
- only Lipschitz continuous decision rules X are considered.

(Bad for applications. Especially the Lipschitz restriction!)

Given a Lipschitz decision rule $X : [0,1] \to \mathcal{X}$,

- type t can 'mimic' $t+m$ by choosing $X(t+m)$
- 'mimicking payoff' $\quad m \mapsto f(X(t+m),t) \quad$ is diff'able a.e.

Classical FOC: $\quad \left.\frac{\mathrm{d}}{\mathrm{d}m} f(X(t+m),t)\right|_{m=0} = 0 \quad$ for a.e. t.

Classical envelope theorem and converse

Classical envelope theorem and converse.
Under the classical assump'ns, classical FOC $\iff$ ⊠ formula.

Proof idea: Differentiation identity for $V_X(t) \coloneqq f(X(t),t)$:

$$V_X'(t) = \underbrace{\frac{\mathrm{d}}{\mathrm{d}m} f(X(t+m),t)\bigg|_{m=0}}_{\text{`indirect effect'}} + \underbrace{f_2(X(t),t)}_{\text{`direct effect'}}.$$

$$\begin{aligned} & V_X'(t) = \text{direct effect} \quad && \text{a.e.} \quad (\boxtimes \text{ formula}) \\ \iff \quad & \text{indirect effect} = 0 && \text{a.e.} \quad (\text{classical FOC}). \end{aligned}$$

Plan

Trouble

Absent classical assumptions,
'mimicking payoff' $m \mapsto f(X(t+m), t)$ need not be diff'able.

– $f(\cdot, t)$ & X need not be diff'able

– actions $\mathcal{X}$ need have no convex or topological structure.

$\implies$ classical FOC ill-defined.

The outer first-order condition

Decision rule X *satisfies the outer FOC* iff

$$\frac{\mathrm{d}}{\mathrm{d}m} \int_r^t f(X(s+m), s)\mathrm{d}s \bigg|_{m=0} = 0 \quad \text{for all } r, t \in (0,1).$$

Motivation: given decision rule $X : [0,1] \to \mathcal{X}$,

– type s can 'mimic' $s+m$ by choosing $X(s+m)$.

– oFOC: if types $s \in [r,t]$ do this,
it's collectively unprofitable (to first order).

Housekeeping I

Housekeeping lemma. Under the classical assump'ns,
oFOC $\iff$ classical FOC.

Sketch proof. Fix a Lipschitz $X : [0,1] \to \mathcal{X}$.

Classical assump'ns & Vitali convergence theorem:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}m} \int_r^t f(X(s+m), s)\mathrm{d}s \bigg|_{m=0}}_{\substack{= 0 \text{ for all } r,t \\ \text{iff oFOC holds}}} = \underbrace{\int_r^t \frac{\mathrm{d}}{\mathrm{d}m} f(X(s+m), s) \bigg|_{m=0} \mathrm{d}s}_{\substack{= 0 \text{ for all } r,t \\ \text{iff classical FOC holds.}}}$$

■

Housekeeping II

Necessity lemma. Any optimal decision rule X satisfies the oFOC & has $V_X(t) \coloneqq f(X(t), t)$ AC.

(sketch proof: slide 46)

Plan

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- Classical envelope theorem and converse

Main theorem
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- Environment and result
- Selling information

Main theorem

Envelope theorem & converse.
For a decision rule $X : [0,1] \to \mathcal{X}$, the following are equivalent:

(1) X satisfies the oFOC

$$\frac{\mathrm{d}}{\mathrm{d}m} \int_r^t f(X(s+m), s)\mathrm{d}s \bigg|_{m=0} = 0 \quad \text{for all } r, t \in (0,1),$$

and $V_X(t) \coloneqq f(X(t), t)$ is absolutely continuous.

(2) X satisfies the ⌧ formula

$$V_X(t) = V_X(0) + \int_0^t f_2(X(s), s)\mathrm{d}s \quad \text{for every } t \in [0,1].$$

Main theorem

Envelope theorem & converse. For $X : [0,1] \to \mathcal{X}$, TFAE:

(1) X satisfies the oFOC, & $V_X(t) \coloneqq f(X(t),t)$ is AC.

(2) X satisfies the $\boxtimes$ formula.

$\Longrightarrow$: an envelope theorem.
Implies the MS02 & classical envelope theorems.

$\Longleftarrow$: *converse* envelope theorem.
Implies the classical converse envelope theorem.

The identity lemma

Proof in classical case relied on differentiation identity

$$V_X'(s) = \underbrace{\frac{\mathrm{d}}{\mathrm{d}m} f(X(s+m),s)\bigg|_{m=0}}_{\text{'indirect effect'}} + \underbrace{f_2(X(s),s)}_{\text{'direct effect'}},$$

or (integrated & rearranged)

$$\int_r^t \frac{\mathrm{d}}{\mathrm{d}m} f(X(s+m),s)\bigg|_{m=0} \mathrm{d}s = V_X(t) - V_X(r) - \int_r^t f_2(X(s),s)\mathrm{d}s.$$

'Outer' version is valid without classical assump'ns:

Identity lemma. If V_X is AC, then

$$\frac{\mathrm{d}}{\mathrm{d}m} \int_r^t f(X(s+m),s)\mathrm{d}s\bigg|_{m=0} = V_X(t) - V_X(r) - \int_r^t f_2(X(s),s)\mathrm{d}s.$$

(Where both sides are well-defined.) (sketch proof: slide 47)

Proof of the main theorem

Identity lemma. If V_X is AC, then

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}m}\int_r^t f(X(s+m),s)\mathrm{d}s\bigg|_{m=0}}_{\substack{=0 \text{ for all } r,t \\ \text{iff oFOC holds.}}} = \underbrace{V_X(t) - V_X(r) - \int_r^t f_2(X(s),s)\mathrm{d}s}_{\substack{=0 \text{ for all } r,t \\ \text{iff } \boxtimes \text{ formula holds.}}}\,.$$

Proof of main theorem. X satisfies the oFOC & V_X is AC
$\implies$ identity lemma applies. So oFOC $\implies$ $\boxtimes$ formula.

X satisfies the $\boxtimes$ formula $\implies$ V_X is AC (by Lebesgue's FToC)
$\implies$ identity lemma applies. So $\boxtimes$ formula $\implies$ oFOC. ∎

Plan

Environment

Agent with preferences $f(y, p, t)$ over physical outcome $y \in \mathcal{Y}$ and payment $p \in \mathbf{R}$.

– $\mathcal{Y}$ partially ordered

– $f(y, \cdot, t)$ strictly decreasing and onto $\mathbf{R}$

– type $t \in [0, 1]$ is agent's private info.

Direct mechanism (Y, P):

– *physical allocation* $Y : [0, 1] \to \mathcal{Y}$

– *payment rule* $P : [0, 1] \to \mathbf{R}$.

IC mechanism: $f(Y(t), P(t), t) \geq f(Y(r), P(r), t) \quad \forall r, t.$

Y is *implementable* iff $\exists$ payment rule $P : [0, 1] \to \mathbf{R}$ s.t. (Y, P) is incentive-compatible.

Goal

'Single-crossing' preferences f:
higher types are more willing to pay to increase $y \in \mathcal{Y}$.

Theorem schema.
If $\mathcal{Y}$ and f are 'regular' & f is 'single-crossing',
then any increasing allocation is implementable.

Versions:

- $\mathcal{Y} \subseteq \mathbf{R}$
  - classical assump'ns — Mirrlees (1976), Spence (1974), Guesnerie and Laffont (1984)
  - no classical assump'ns — Nöldeke and Samuelson (2018)
- general $\mathcal{Y}$
  - quasi-linear f — Matthews and Moore (1987), García (2005)
  - general f — this paper.

Regularity and single-crossing

Regular $\mathcal{Y}$: ‘rich’ & ‘not too large’. (def’n: slide 50)

Examples:

- $\mathbf{R}^n$ ordered by ‘coordinate-wise smaller’
- finite-expectation RVs ordered by ‘a.s. smaller’
- distributions of posteriors updated from a given prior ordered by Blackwell.

Regular f: (def’n: slide 51)

(a) type derivative f_3 exists, bounded, continuous in p.

(b) f jointly continuous (when $\mathcal{Y}$ has order topology).

Single-crossing f: (def’n: slide 52)
if type t willing to pay to increase $y \in \mathcal{Y}$, then so is type $t' > t$.

Implementability theorem.
If $\mathcal{Y}$ and f are regular & f is single-crossing,
then any increasing physical allocation is implementable.

Argument I

Fix an increasing physical allocation $Y : [0,1] \to \mathcal{Y}$.

Choose a payment rule P so that $\boxtimes$ holds.

- $\boxtimes$ formula is integral eq'n in unknown $P : [0,1] \to \mathbf{R}$:

$$\phi(P(t), t) = \phi(P(0), 0) + \int_0^t \psi(P(s), s) \mathrm{d}s$$

where $\phi(p,t) \coloneqq f(Y(t), p, t) \quad \& \quad \psi(p,t) \coloneqq f_3(Y(t), p, t)$

- f regular $\implies$ solution exists.

Argument II

Embed into general setting: $\begin{cases} x \;:=\; (y,p) \;\in\; \mathcal{Y} \times \mathbf{R} \;=:\; \mathcal{X} \\ f(x,t) \;=\; f(y,p,t) \\ \text{mechanisms } X \;=\; (Y,P). \end{cases}$

f regular $\implies$ type derivative exists & is bounded.

By *converse envelope theorem,* $\boxtimes \implies$ oFOC
$\iff$ mechanism (Y,P) is locally IC.

Finally, local IC $\implies$ global IC by single-crossing.

Plan

Selling information

Monopolist selling information.

Physical allocations $\mathcal{Y}$:
distributions of posterior beliefs, ordered by Blackwell.

Selling information: model

Finite set Ω of states. Prior $\mu_0 \in \operatorname{int} \Delta(\Omega)$.

Distribution of posteriors: a Borel probability y on $\Delta(\Omega)$.

Fact: y is feasible iff $\int_{\Delta(\Omega)} \mu y(\mathrm{d}\mu) = \mu_0$.

Physical outcomes $\mathcal{Y}$: dist'ns of posteriors with mean μ_0.

Equip $\mathcal{Y}$ with Blackwell order: $y \lesssim y'$ iff

$$\int_{\Delta(\Omega)} v \mathrm{d}y \leq \int_{\Delta(\Omega)} v \mathrm{d}y' \qquad \forall \text{ cont's convex } v : \Delta(\Omega) \to \mathbf{R}.$$

$\mathcal{Y}$ is regular.

Selling information: preferences

Agent's value of information: $V(\mu, t)$.

- V convex in μ
- V_2 exists, bounded, cont's in μ
- micro-foundation: $V(\mu, t) = \sup_{a \in A} \sum_{\omega \in \Omega} U(a, \omega, t)\mu(\omega)$.

Agent's preferences: $f(y, p, t) \coloneqq g\Big(\int_{\Delta(\Omega)} V(\mu, t) y(\mathrm{d}\mu), p\Big)$.

Assume single-crossing: higher types more willing
to pay for extra information.

Selling information: implementability

Information allocation: $Y : [0,1] \to \mathcal{Y}$.

Implementability theorem $\implies$ any Blackwell-increasing information allocation is implementable.

Thanks!

Absolute equi-continuity: properties

A family $\{\phi_x\}_{x\in\mathcal{X}}$ of functions $[0,1] \to \mathbf{R}$ is *AEC* iff the family

$$\left\{ t \mapsto \sup_{x\in\mathcal{X}} \left| \frac{\phi_x(t+m) - \phi_x(t)}{m} \right| \right\}_{m>0} \qquad \text{is uniformly integrable.}$$

Name inspired by the following (Fitzpatrick & Hunt, 2015):

AC–UI lemma. A continuous $\phi : [0,1] \to \mathbf{R}$ is AC iff

$$\left\{ t \mapsto \frac{\phi(t+m) - \phi(t)}{m} \right\}_{m>0} \qquad \text{is uniformly integrable.}$$

As name 'AEC' suggests, an AEC family

– is (uniformly) equi-continuous

– has AC functions as its members.

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Sketch proof of the necessity lemma

Necessity lemma. Any optimal decision rule X satisfies the oFOC & has $V_X(t) \coloneqq f(X(t),t)$ AC.

Sketch proof. X optimal & $\{f(x,\cdot)\}_{x\in\mathcal{X}}$ AEC $\implies$ V_X AC.

Since X optimal, have for any s and $m > 0 > m'$ that

$$\frac{f(X(s+m),s) - f(X(s),s)}{m} \leq 0 \leq \frac{f(X(s+m'),s) - f(X(s),s)}{m'}.$$

Integrating over (r,t) and letting $m, m' \to 0$,

both sides (in fact) converge to same limit:

$$\frac{\mathrm{d}}{\mathrm{d}m} \int_r^t f(X(s+m),s)\mathrm{d}s\bigg|_{m=0} \leq 0 \leq \frac{\mathrm{d}}{\mathrm{d}m} \int_r^t f(X(s+m),s)\mathrm{d}s\bigg|_{m=0}. \blacksquare$$

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Sketch proof of the identity lemma I

For $m > 0$, write

$$\left.\begin{aligned}
&\frac{V_X(t+m)-V_X(t)}{m}\right\} =: \phi_m(t) \\
= \quad &\left.\frac{f(X(t+m),t+m)-f(X(t+m),t)}{m}\right\} =: \psi_m(t) \\
+ \quad &\left.\frac{f(X(t+m),t)-f(X(t),t)}{m}\right\} =: \chi_m(t).
\end{aligned}$$

$$\lim_{m\downarrow 0}\int_r^t \chi_m = \frac{\mathrm{d}}{\mathrm{d}m}\int_r^t f(X(s+m),s)\mathrm{d}s\Big|_{m=0} \qquad \text{if limit exists.}$$

Must show: limit exists & equals

$$V_X(t)-V_X(r)-\int_r^t f_2(X(s),s)\mathrm{d}s.$$

Sketch proof of the identity lemma II

$$
\begin{aligned}
&\left.\frac{V_X(t+m)-V_X(t)}{m}\right\} =: \phi_m(t)\\
=\ &\left.\frac{f(X(t+m),t+m)-f(X(t+m),t)}{m}\right\} =: \psi_m(t)\\
+\ &\left.\frac{f(X(t+m),t)-f(X(t),t)}{m}\right\} =: \chi_m(t).
\end{aligned}
$$

$\{\psi_m\}_{m>0}$ need not converge a.e. (Even with strong assump'ns.)

But consider $\psi_m^\star(t) := \dfrac{f(X(t),t)-f(X(t),t-m)}{m}$.

$\{\psi_m^\star\}_{m>0}$ is UI & converges pointwise to $t \mapsto f_2(X(t),t)$. And

$$
\begin{aligned}
\int_r^t \psi_m = \int_{r+m}^{t+m} \psi_m^\star &= \int_r^t \psi_m^\star + \left(\int_t^{t+m} \psi_m^\star - \int_r^{r+m} \psi_m^\star\right)\\
&= \int_r^t \psi_m^\star + \mathrm{o}(1) \qquad \text{by UI.}
\end{aligned}
$$

Sketch proof of the identity lemma III

$$\left.\frac{V_X(t+m)-V_X(t)}{m}\right\} =: \phi_m(t)$$

$$= \left.\frac{f(X(t+m),t+m)-f(X(t+m),t)}{m}\right\} =: \psi_m(t)$$

$$+ \left.\frac{f(X(t+m),t)-f(X(t),t)}{m}\right\} =: \chi_m(t).$$

$\int_r^t \psi_m = \int_r^t \psi_m^\star + \mathrm{o}(1)$, $\{\psi_m^\star\}_{m>0}$ UI & converges pointwise to $t \mapsto f_2(X(t),t)$.

V_X AC $\implies$ $\{\phi_m\}_{m>0}$ UI & converges a.e. to V_X'. So

$$\lim_{m\downarrow 0}\int_r^t \chi_m = \lim_{m\downarrow 0}\int_r^t [\phi_m - \psi_m] = \lim_{m\downarrow 0}\int_r^t [\phi_m - \psi_m^\star]$$

$$\text{(Vitali)} \quad = \int_r^t \lim_{m\downarrow 0}[\phi_m - \psi_m^\star] = \int_r^t \left[V_X'(s) - f_2(X(s),s)\right]\mathrm{d}s$$

$$\text{(FToC)} \quad = V_X(t) - V_X(r) - \int_r^t f_2(X(s),s)\mathrm{d}s.$$

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Outcome regularity

A set $\mathcal{A}$ partially ordered by $\lesssim$ is

(1) *order-dense-in-itself* iff for any $a < a'$ in $\mathcal{A}$,
there is a $b \in \mathcal{A}$ such that $a < b < a'$,

(2) *chain-separable* iff for each chain $C \subseteq \mathcal{A}$,
there is a countable set $B \subseteq \mathcal{A}$ that is order-dense in C,‡

(3) *countably chain-complete* iff every countable chain in $\mathcal{A}$
with a lower (upper) bound in $\mathcal{A}$
has an infimum (a supremum) in $\mathcal{A}$.

(1) & (2): $\mathcal{A}$ 'rich'. (3): $\mathcal{A}$ 'not too large'.

Definition. $\mathcal{Y}$ is *regular* iff it satisfies properties (1)–(3).

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‡ $B \subseteq \mathcal{A}$ is *order-dense* iff for any $a < a'$ in $\mathcal{A}$, $\exists\, b \in B$ s.t. $a \lesssim b \lesssim a'$.

Preference regularity

Order topology on a set $\mathcal{A}$ partially ordered by $\lesssim$:
the topology generated by the open order rays

$$\{b \in \mathcal{A} : b < a\} \quad \text{and} \quad \{b \in \mathcal{A} : a < b\}.$$

Definition. f is *regular* iff

(a) type derivative f_3 exists & is bounded & continuous in p

(b) for any chain $\mathcal{C} \subseteq \mathcal{Y}$, f jointly continuous on $\mathcal{C} \times \mathbf{R} \times [0,1]$ when $\mathcal{C}$ has relative top'gy inherited from order top'gy on $\mathcal{Y}$.

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Single-crossing

Definition. For $\phi : [0,1] \to \mathbf{R}$, upper & lower derivatives

$$\mathrm{D}^\star \phi(t) := \limsup_{m \to 0} \frac{\phi(t+m) - \phi(t)}{m}$$
$$\mathrm{D}_\star \phi(t) := \liminf_{m \to 0} \frac{\phi(t+m) - \phi(t)}{m}.$$

Partial upper/lower derivatives: $(\mathrm{D}^\star)_i$ & $(\mathrm{D}_\star)_i$.

Definition. f is *single-crossing* iff
for any increasing $Y : [0,1] \to \mathcal{Y}$ & any $P : [0,1] \to \mathbf{R}$,
mis-reporting payoff $U(r,t) := f(Y(r), P(r), t)$ satisfies

$$\begin{aligned} &(\mathrm{D}^\star)_1 U(t,t) \geq 0 \quad \text{implies} \quad (\mathrm{D}_\star)_1 U(t,t') > 0 \quad \text{for } t' > t \\ \text{and} \quad &(\mathrm{D}_\star)_1 U(t,t) \leq 0 \quad \text{implies} \quad (\mathrm{D}^\star)_1 U(t,t') < 0 \quad \text{for } t' < t. \end{aligned}$$

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