

THE CONVERSE ENVELOPE THEOREM

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paper: [arXiv.org/abs/1909.11219](https://arxiv.org/abs/1909.11219)

Envelope theorem: optimal decision-making \implies \boxtimes formula.

Textbook intuition: \boxtimes formula is consequence of FOC.

Modern envelope theorem of MS02:¹ almost no assumptions.

\leftrightarrow FOC ill-defined, so need different intuition.

My theorem: with almost no assumptions,
 \boxtimes formula equivalent to generalised FOC.

– an envelope theorem: FOC \implies \boxtimes

– a *converse*: $\boxtimes \implies$ FOC.

Application to mechanism design.

¹Milgrom, P., & Segal, I. (2002). Envelope theorems for arbitrary choice sets. *Econometrica*, 70(2), 583–601. doi:10.1111/1468-0262.00296

Environment

Agent chooses action x from a set \mathcal{X}
to maximise objective $f(x, t)$, where $t \in [0, 1]$ is a parameter.

No assumptions on \mathcal{X} , almost none on f :

(1) $f(x, \cdot)$ is differentiable for each $x \in \mathcal{X}$

(2) $f(x, \cdot)$ is ‘not too erratic’. (definition: slide 12)

Decision rule: a map $X : [0, 1] \rightarrow \mathcal{X}$.

Associated value function: $V_X(t) := f(X(t), t)$.

Envelope theorem

X satisfies the \boxtimes formula iff

$$V_X(t) = V_X(0) + \int_0^t f_2(X(s), s) ds \quad \text{for every } t \in [0, 1].$$

Equivalently: V_X is absolutely continuous and

$$V_X'(t) = f_2(X(t), t) \quad \text{for a.e. } t \in [0, 1].$$

X is optimal iff for every t , $X(t)$ maximises $f(\cdot, t)$.

Modern envelope theorem (MS02).

Any optimal decision rule satisfies the \boxtimes formula.

Textbook intuition

Differentiation identity:

$$V'_X(t) = \underbrace{\frac{d}{dm} f(X(t+m), t) \Big|_{m=0}}_{\text{'indirect effect'}} + \underbrace{f_2(X(t), t)}_{\text{'direct effect'}}.$$

$$\begin{aligned} & V'_X(t) = \text{direct effect} && (\boxtimes \text{ formula}) \\ \iff & \text{indirect effect} = 0 && (\text{FOC}). \end{aligned}$$

Problem: 'indirect effect' (hence FOC) ill-defined!

- $f(\cdot, t)$ & X need not be differentiable.
- actions \mathcal{X} need have no convex or topological structure.

The outer first-order condition

Disjuncture: in general, \boxtimes formula $\not\leftrightarrow$ FOC.

- one solution: add strong ‘classical’ assumptions. (slide 13)
- my solution: find the correct FOC!

Decision rule X satisfies the outer FOC iff

$$\frac{d}{dm} \int_r^t f(X(s+m), s) ds \Big|_{m=0} = 0 \quad \text{for all } r, t \in (0, 1).$$

‘Integrated’ version of classical FOC.

- always well-defined
- equiv’nt to classical FOC when latter well-defined. (slide 13)

Theorem

Envelope theorem & converse.

For a decision rule $X : [0, 1] \rightarrow \mathcal{X}$, the following are equivalent:

(1) X satisfies the oFOC

$$\frac{d}{dm} \int_r^t f(X(s+m), s) ds \Big|_{m=0} = 0 \quad \text{for all } r, t \in (0, 1),$$

and $V_X(t) := f(X(t), t)$ is absolutely continuous.

(2) X satisfies the \boxtimes formula

$$V_X(t) = V_X(0) + \int_0^t f_2(X(s), s) ds \quad \text{for every } t \in [0, 1].$$

(proof idea: slide 14)

Mechanism design application: environment

Agent with preferences $f(y, p, t)$ over physical outcome $y \in \mathcal{Y}$ and payment $p \in \mathbf{R}$.

- type $t \in [0, 1]$ is agent's private info
- assume single-crossing.

What's new:

- outcome space \mathcal{Y} is an abstract partially ordered set
- preferences not assumed quasi-linear in payment.

A *physical allocation* is $Y : [0, 1] \rightarrow \mathcal{Y}$.

Y is *implementable* iff \exists payment rule $P : [0, 1] \rightarrow \mathbf{R}$
s.t. (Y, P) is incentive-compatible.

$$\left(\text{viz. } f(Y(t), P(t), t) \geq f(Y(r), P(r), t) \quad \text{for all } r, t. \right)$$

Mechanism design application: theorem

Implementability theorem.

Under weak regularity assumptions,
any increasing physical allocation is implementable.

Argument:

- fix an increasing physical allocation $Y : [0, 1] \rightarrow \mathcal{Y}$
- choose a payment rule P so that \boxtimes holds
- then by *converse envelope theorem*, oFOC holds
 \iff mechanism (Y, P) is locally IC.
- finally, local IC \implies global IC by single-crossing.

Mechanism design application: example

Monopolist selling information.

Physical allocations \mathcal{Y} :

distributions of posterior beliefs, ordered by Blackwell.

By the implementability theorem, any information allocation that gives higher types Blackwell-better signals can be implemented.

Thanks!



Definition of ‘not too erratic’

A family $\{\phi_x\}_{x \in \mathcal{X}}$ of functions $[0, 1] \rightarrow \mathbf{R}$ is *uniformly absolutely continuous (UAC)* iff the family

$$\left\{ t \mapsto \sup_{x \in \mathcal{X}} \left| \frac{\phi_x(t+m) - \phi_x(t)}{m} \right| \right\}_{m>0}$$

is uniformly integrable.

‘ $f(x, \cdot)$ not too erratic’ (slide 3)

means precisely that $\{f(x, \cdot)\}_{x \in \mathcal{X}}$ is UAC.

- a sufficient condition (maintained by MS02):
 $t \mapsto \sup_{x \in \mathcal{X}} |f_2(x, t)|$ dominated by an integrable function.
- a stronger sufficient condition: f_2 bounded.

↔ back to environment (slide 3)

Classical assumptions

Classical assumptions:

- \mathcal{X} is a convex subset of \mathbf{R}^n
- action derivative f_1 exists & is bounded
- only Lipschitz continuous decision rules X are considered.

(Bad for applications. Especially the Lipschitz restriction!)

Classical FOC: $\left. \frac{d}{dm} f(X(t+m), t) \right|_{m=0} = 0$ for a.e. t .

Classical envelope theorem and converse.

Under the classical assump'ns, classical FOC \iff \boxtimes formula.

Housekeeping lemma.

oFOC \iff classical FOC whenever the latter is well-defined.

\hookrightarrow back to oFOC (slide 6)

Proof idea

Textbook intuition was based on differentiation identity:

$$V'_X(s) = \underbrace{\frac{d}{dm} f(X(s+m), s) \Big|_{m=0}}_{\text{'indirect effect'}} + \underbrace{f_2(X(s), s)}_{\text{'direct effect'}}$$

or (integrating)

$$V_X(t) - V_X(r) = \int_r^t \frac{d}{dm} f(X(s+m), s) \Big|_{m=0} ds + \int_r^t f_2(X(s), s) ds.$$

I prove that the 'outer' version is always valid:

$$V_X(t) - V_X(r) = \underbrace{\frac{d}{dm} \int_r^t f(X(s+m), s) ds \Big|_{m=0}}_{\text{'indirect effect'}} + \underbrace{\int_r^t f_2(X(s), s) ds}_{\text{'direct effect'}}.$$

The rest is easy:

$$\begin{aligned} V_X(t) - V_X(r) &= \text{direct effect} && (\boxtimes \text{ formula}) \\ \iff \text{indirect effect} &= 0 && (\text{oFOC}). \end{aligned}$$