Screening for breakthroughs: Addendum on comparative statics

Gregorio Curello
University of Bonn

Ludvig Sinander
Northwestern University

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In this addendum to Curello and Sinander (2020), we prove that as the breakthrough distribution G becomes later in the sense of monotone likelihood ratio (MLR), the optimal disclosure reward X increases pointwise.

Comparative statics theorem. Suppose that F^0 is strictly concave and that the right-derivatives of F^0, F^1 at $u = 0$ are finite. Let $G, G^\dagger$ be absolutely continuous distributions with equal, unbounded support and $G(0) = G^\dagger(0) = 0$. If G MLR-dominates $G^\dagger$,[1] then $X \geq X^\dagger$ for any mechanisms (x, X) and $(x^\dagger, X^\dagger)$ that are optimal for G and $G^\dagger$, respectively.

The restriction to absolutely continuous distributions $G, G^\dagger$ with equal support is merely for simplicity. We prove the theorem in §2 below, relying on a number of intermediate results stated in the next section. The remainder of this document is devoted to proving these intermediate results.

1 Lemmata

Definition 1. F^0, F^1 are *simple* if they are strictly concave and continuously differentiable, $F^{1\prime}$ is Lipschitz continuous on $[u^\star, u^0]$, and $u^\star > 0$.

Definition 2. Suppose that F^0, F^1 are simple and that G has finite support, enumerated $\text{supp}(G) = \{t_k\}_{k=1}^K$ for $K \in \mathbf{N}$, where $0 \leq t_1 \leq \cdots \leq t_K < \infty$. Say that (x, X) is *simply optimal for* (F^0, F^1, G) iff

$$x_t = \begin{cases} u^0 & \text{for } t \in [0, t_1) \\ x_{t_k} & \text{for } t \in [t_k, t_{k+1}) \text{ for some } k \in \{1, \dots, K-1\} \\ x_{t_K} & \text{for } t \in [t_K, \infty) \end{cases} \tag{S}$$

[1] G MLR-dominates $G^\dagger$ iff the ratio $t \mapsto G'(t) / G^{\dagger\prime}(t)$ of their densities is well-defined and increasing on the support.

for a decreasing sequence $(x_{t_k})_{k=1}^K$ in $[u^\star, u^0]$ such that

$$\mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big) = 0$$
$$\text{and} \quad F^{0\prime}(x_{t_k}) = \mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau > t_k\Big) \quad \text{for all } k \in \{1, \dots, K-1\}.$$

Remark 1. The 'optimal' in 'simply optimal' is motivated by the fact that if (x, X) is simply optimal for simple F^0, F^1 and finite-support G, then it satisfies the Euler equation (as defined in appendix E, p. 41 of the paper). Thus provided (x, X) is undominated, it is optimal by the Euler lemma (appendix E, p. 41).

Lemma 1. Suppose that F^0 is strictly concave and that the right-derivatives of F^0, F^1 at $u = 0$ are finite, and let G have unbounded support. Then if a mechanism satisfies the Euler equation, it is uniquely optimal for G.

The proof is in §3. We prove the following two propositions in §4–5 below:

Proposition 1. If F^0, F^1 are simple and G has finite support, then there exists a mechanism (x, X) that is simply optimal for (F^0, F^1, G).

Proposition 2. Suppose that F^0, F^1 are simple. Let $G, G^\dagger$ be finite-support distributions with $G(0) = G^\dagger(0) = 0$ and equal support. If G MLR-dominates $G^\dagger$,[2] then $X \geq X^\dagger$ for any mechanisms (x, X) and $(x^\dagger, X^\dagger)$ that are simply optimal for G and $G^\dagger$, respectively.

For the next result, let $\mathcal{X}'$ be the set of decreasing (up to a.e. equivalence) maps $x : \mathbf{R}_+ \to [u^\star, u^0]$ equipped with the norm

$$\|x\| := \int_0^\infty re^{-rt}|x_t|\mathrm{d}t.$$

Lemma 2. $\mathcal{X}'$ is compact.

This sounds standard, but we have been unable to locate a reference, so provide a proof in §6 below. The next result is proved in §7:

Lemma 3. Suppose that F^0, F^1 are simple. Let $(G_n)_{n\in\mathbf{N}}$ be a sequence of finite-support CDFs with $G_n(0) = 0$ converging weakly to G. Let $(x^n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{X}'$ along which (x^n, X^n) is simply optimal for (F_n^0, F_n^1, G_n) for each $n \in \mathbf{N}$, and suppose that $x^n \to x \in \mathcal{X}'$ a.e. Then (x, X) satisfies the Euler equation for (F^0, F^1, G) (defined in appendix E, p. 41 of the paper).

[2] In the finite-support case, G MLR-dominates $G^\dagger$ iff the ratio $t \mapsto g(t)/g^\dagger(t)$ of their probability mass functions is increasing on the support.

Finally, we prove the following in §8:

Proposition 3. Suppose that the right-derivatives of F^0, F^1 at $u = 0$ are finite. Then there exists a sequence $\left(F_n^0, F_n^1\right)_{n\in\mathbf{N}}$ of simple technologies such that for any CDF G with $G(0) = 0$, a mechanism x satisfies the Euler equation for $\left(F^0, F^1, G\right)$ if there is a sequence $(x^n)_{n\in\mathbf{N}}$ in $\mathcal{X}'$ that converges a.e. to x, and along which (x^n, X^n) satisfies the Euler equation for $\left(F_n^0, F_n^1, G\right)$ for each $n \in \mathbf{N}$.

2 Proof of the comparative statics theorem

Assume that the right-derivatives of F^0, F^1 at $u = 0$ are finite, and let $G, G^\dagger$ be absolutely continuous with $G(0) = G^\dagger(0) = 0$ and equal support such that the former MLR-dominates the latter. Let $\left(F_n^0, F_n^1\right)_{n\in\mathbf{N}}$ be simple technologies satisfying the hypothesis of Proposition 3. Clearly there exist sequences $(G_n)_{n\in\mathbf{N}}$ and $(G_n^\dagger)_{n\in\mathbf{N}}$ of finite-support CDFs converging weakly to (respectively) G and $G^\dagger$ such that for each $n \in \mathbf{N}$, $G_n, G_n^\dagger$ have equal support, $G_n(0) = G_n^\dagger(0) = 0$, and G_n MLR-dominates $G_n^\dagger$.[3]

Fix an arbitrary $n \in \mathbf{N}$. By Proposition 1, there exist for every $m \in \mathbf{N}$ mechanisms (x^{nm}, X^{nm}) and $(x^{\dagger,nm}, X^{\dagger,nm})$ that are simply optimal for $\left(F_n^0, F_n^1, G_m\right)$ and $\left(F_n^0, F_n^1, G_m^\dagger\right)$, respectively. By definition of 'simply optimal', x^{nm} and $x^{\dagger,nm}$ belong to $\mathcal{X}'$.

By Lemma 2, $\mathcal{X}' \times \mathcal{X}'$ is compact in the product topology. Hence we may assume (passing to a subsequence if necessary) that

$$\lim_{m\to\infty} x^{nm} = x^n \text{ a.e.} \quad \text{and} \quad \lim_{m\to\infty} x^{\dagger,nm} = x^{\dagger,n} \text{ a.e.} \tag{1}$$

[3] The sequences can be constructed as follows. Let $\{q_n\}_{n\in\mathbf{N}\cup\{0\}}$ be an enumeration of $\operatorname{supp}(G) \cap \mathbf{Q}$, where $q_0 = \min \operatorname{supp}(G)$ and q_1 is such that $G(q_1) > 0$ and $G^\dagger(q_1) > 0$. For all $n \in \mathbf{N}$, express $\{q_k\}_{k=0}^n = \{q_k^n\}_{k=0}^n$ where $q_0^n < \cdots < q_n^n$, and define G and $G^\dagger$ by

$$G_n^{(\dagger)}(t) = \frac{1}{G^{(\dagger)}(q_n^n)} \sum_{k=1}^n \mathbf{1}_{\left[0,q_k^n\right]}(t) \left[G^{(\dagger)}(q_k^n) - G^{(\dagger)}(q_{k-1}^n)\right] \quad \text{for each } t \in \mathbf{R}_+.$$

Then G_n and $G_n^\dagger$ both have support $\{q_k\}_{k=1}^n$ since $G(0) = G^\dagger(0) = 0$. To prove that G_n MLR-dominates $G_n^\dagger$, note that this is immediate for $n = 1$, so fix $n \geq 2$. Let g ($g^\dagger$) be the density of G ($G^\dagger$) with respect to the Lebesgue measure, and g_n ($g_n^\dagger$) the probability mass function of G_n ($G_n^\dagger$). For $1 \leq k < n$, we have

$$\frac{g_n\left(q_{k+1}^n\right)}{g_n^\dagger\left(q_{k+1}^n\right)} = \frac{G_n^\dagger(q_n^n)}{G_n(q_n^n)} \frac{\int_{q_k^n}^{q_{k+1}^n} g}{\int_{q_k^n}^{q_{k+1}^n} g^\dagger} \geq \frac{G_n^\dagger(q_n^n)}{G_n(q_n^n)} \frac{g(q_k^n)}{g^\dagger(q_k^n)} \geq \frac{G_n^\dagger(q_n^n)}{G_n(q_n^n)} \frac{\int_{q_{k-1}^n}^{q_k^n} g}{\int_{q_{k-1}^n}^{q_k^n} g^\dagger} = \frac{g_n(q_k^n)}{g_n^\dagger(q_k^n)},$$

where the inequalities hold since $t \mapsto g(t)/g^\dagger(t)$ is (well-defined and) increasing on $\operatorname{supp}(G)$.

for some $x^n, x^{\dagger,n} \in \mathcal{X}'$. Similarly (again passing to a subsequence if required),

$$\lim_{n\to\infty} x^n = x \text{ a.e.} \quad \text{and} \quad \lim_{n\to\infty} x^{\dagger,n} = x \text{ a.e.} \tag{2}$$

for some $x, x^\dagger \in \mathcal{X}'$. It suffices to show (i) that the mechanism (x, X) is uniquely optimal for G, (ii) that $(x^\dagger, X^\dagger)$ is uniquely optimal for $G^\dagger$, and (iii) that $X \geq X^\dagger$.

For (i), it suffices by Lemma 1 to show that (x, X) satisfies the Euler equation. For any given $n \in \mathbf{N}$, (x^n, X^n) satisfies the Euler equation for (F^0_n, F^1_n, G) by Lemma 3. It follows from Proposition 3 that (x, X) satisfies the Euler equation for (F^0, F^1, G), as desired. Part (ii) follows from an analogous argument.

For (iii), Proposition 2 provides that $X^{nm} \geq X^{\dagger,nm}$ for all $n, m \in \mathbf{N}$. By (1), fixing an arbitrary $n \in \mathbf{N}$ and letting $m \to \infty$ yields $X^n \geq X^{\dagger,n}$. By (2), letting $n \to \infty$ yields $X \geq X^\dagger$, as desired. ■

3 Proof of Lemma 1

Write $\mathcal{X}$ for the space of measurable functions $\mathbf{R}_+ \to [0, u^0]$, and note that it is convex. For a given distribution G, and define $\pi_G : \mathcal{X} \to \mathbf{R}$ by

$$\pi_G(x) \coloneqq \Pi_G(x, X) = \mathbf{E}_G\left(\int_0^\tau re^{-rs} F^0(x_s)\mathrm{d}s + e^{-r\tau} F^1(X_\tau)\right).$$

Observation 1. If F^0 is strictly concave and G has unbounded support, then at most one mechanism maximises π_G.

Proof of Lemma 1. Assume that F^0, F^1 and G satisfy the hypothesis, and let (x, X) satisfy the Euler equation. By Observation 1 and by Proposition 5 in supplemental appendix K (p. 71 in the paper), there is exactly one mechanism $x \in \mathcal{X}$ that maximises π_G. It is undominated since $\pi_G(x) > \pi_G(x^\dagger)$ for any other mechanism $x^\dagger$. ■

Proof of Observation 1. It suffices to show that π_G is strictly concave. To that end, fix distinct $x, x^\dagger$ (i.e. $x \neq x^\dagger$ on a non-null set) and $\lambda \in (0, 1)$. Observe that

$$F^0\left(\lambda x + (1-\lambda)x^\dagger\right) \geq \lambda F^0(x) + (1-\lambda)F^0\left(x^\dagger\right)$$
$$\text{and} \quad F^1\left(\lambda X + (1-\lambda)X^\dagger\right) \geq \lambda F^1(X) + (1-\lambda)F^1\left(X^\dagger\right)$$

by concavity of F^0, F^1, and that the former inequality is strict on a non-null set by strict concavity of F^0. Since G has unbounded support, it follows that

$$\begin{aligned}
&\pi_G\left(\lambda x + (1-\lambda)x^\dagger\right) \\
&= \mathbf{E}_G\left(r\int_0^\tau e^{-rt}F^0\left(\lambda x_t + (1-\lambda)x_t^\dagger\right) + e^{-r\tau}F^1\left(\lambda X_\tau + (1-\lambda)X_\tau^\dagger\right)\right) \\
&> \mathbf{E}_G\left(r\int_0^\tau e^{-rt}\left[\lambda F^0(x_t) + (1-\lambda)F^0\left(x_t^\dagger\right)\right]\right. \\
&\quad \left. + e^{-r\tau}\left[\lambda F^1(X_\tau) + (1-\lambda)F^1\left(X_\tau^\dagger\right)\right]\right) \\
&= \lambda\pi_G(x) + (1-\lambda)\pi_G\left(x^\dagger\right).
\end{aligned}$$

■

4 Proof of Proposition 2

Fix mechanisms (x, X) and $(x^\dagger, X^\dagger)$ that are simply optimal for (F^0, F^1, G) and $(F^0, F^1, G^\dagger)$, respectively; we must show that $X \geq X^\dagger$. Enumerate the support of G and $G^\dagger$ as $\{t_k\}_{k=1}^K \subseteq \mathbf{R}_+$, where $K \in \mathbf{N}$ and

$$0 \leq t_1 < \cdots < t_K < \infty.$$

Claim. It suffices to show that $X_{t_k} \geq X_{t_k}^\dagger$ for every $k \in \{1, \ldots, K\}$.

Proof. Suppose that $X_{t_k} \geq X_{t_k}^\dagger$ for every $k \in \{1, \ldots, K\}$, and fix an arbitrary $t \in \mathbf{R}_+$; we shall show that $X_t \geq X_t^\dagger$. If $t \geq t_K$, then

$$X_t = X_{t_K} \geq X_{t_K}^\dagger = X_t^\dagger$$

by (S). Assume for the remainder that $t < t_K$.

Suppose that, for some $k \leq K$, we have $t \leq t_k$ and $x^\dagger \leq x$ on (t, t_k). (Note that this holds if $t \leq t_1$, since $x^\dagger \leq u^0 = x$ on $[0, t_1)$ by (S).) Then

$$\begin{aligned}
X_t - X_t^\dagger = \int_t^{t_k} re^{-r(s-t)}\left(x_s - x_s^\dagger\right)\mathrm{d}s + e^{-r(t_k-t)}\left(X_{t_k} - X_{t_k}^\dagger\right)& \\
\geq e^{-r(t_k-t)}\left(X_{t_k} - X_{t_k}^\dagger\right) \geq 0.&
\end{aligned}$$

Suppose instead that $t \in (t_k, t_{k+1})$ for some $k < K$ and that $x_s^\dagger > x_s$ for some $s \in (t_k, t_{k+1})$. Then we have $x^\dagger > x$ on (t_k, t) by (S), so that

$$\begin{aligned}
0 \leq X_{t_k} - X_{t_k}^\dagger = \int_{t_k}^t re^{-r(s-t_k)}\left(x_s - x_s^\dagger\right)\mathrm{d}s + e^{-r(t-t_k)}\left(X_t - X_t^\dagger\right)& \\
\leq e^{-r(t-t_k)}\left(X_t - X_t^\dagger\right).&
\end{aligned}$$

□

To show that $X_{t_k} \geq X^\dagger_{t_k}$ for every $k \in \{1, \ldots, K\}$, suppose not; we shall derive a contradiction. Let k' denote the largest $k \in \{1, \ldots, K\}$ at which $X_{t_k} < X^\dagger_{t_k}$. We shall prove that for every $k \leq k'$, it holds that

$$X_{t_k} < X^\dagger_{t_k} \tag{3}$$

$$\mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau \geq t_k\Big) > \mathbf{E}_{G^\dagger}\Big(F^{1\prime}\Big(X^\dagger_\tau\Big)\Big|\tau \geq t_k\Big). \tag{4}$$

This suffices because it contradicts the fact that

$$\begin{aligned}\mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau \geq t_1\Big) &= \mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big) \\ &= 0 = \mathbf{E}_{G^\dagger}\Big(F^{1\prime}\Big(X^\dagger_\tau\Big)\Big) = \mathbf{E}_{G^\dagger}\Big(F^{1\prime}\Big(X^\dagger_\tau\Big)\Big|\tau \geq t_1\Big).\end{aligned}$$

We proceed by induction on $k \in \{k', \ldots, 1\}$.

Base case: $k = k'$. Here (3) holds by hypothesis, so we need only derive (4). If $k' = K$, then we have by strict concavity of F^1 that

$$\mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau \geq t_k\Big) = F^{1\prime}(X_{t_K}) > F^{1\prime}\Big(X^\dagger_{t_K}\Big) = \mathbf{E}_{G^\dagger}\Big(F^{1\prime}\Big(X^\dagger_\tau\Big)\Big|\tau \geq t_k\Big).$$

Assume for the remainder that $k' < K$.

Since $X_{t_k} < X^\dagger_{t_k}$ and $X_{t_{k+1}} \geq X^\dagger_{t_{k+1}}$ by definition of k', (S) yields

$$\begin{aligned}\Big(1 - e^{-r(t_{k+1}-t_k)}\Big)x_{t_k} &= X_{t_k} - e^{-r(t_{k+1}-t_k)}X_{t_{k+1}} \\ &< X^\dagger_{t_k} - e^{-r(t_{k+1}-t_k)}X^\dagger_{t_{k+1}} = \Big(1 - e^{-r(t_{k+1}-t_k)}\Big)x^\dagger_{t_k},\end{aligned}$$

so that $x_{t_k} < x^\dagger_{t_k}$. It follows by strict concavity of F^0 that

$$\mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau > t_k\Big) = F^{0\prime}(x_{t_k}) > F^{0\prime}\Big(x^\dagger_{t_k}\Big) = \mathbf{E}_{G^\dagger}\Big(F^{1\prime}\Big(X^\dagger_\tau\Big)\Big|\tau > t_k\Big),$$

which is to say that (4) holds at $k+1$. Thus (4) holds at k:

$$\begin{aligned}\mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau \geq t_k\Big) &= \mathbf{P}_G(\tau = t_k|\tau \geq t_k)F^{1\prime}(X_{t_k}) \\ &\quad + \mathbf{P}_G(\tau > t_k|\tau \geq t_k)\mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau > t_k\Big) \\ &\geq \mathbf{P}_{G^\dagger}(\tau = t_k|\tau \geq t_k)F^{1\prime}(X_{t_k}) \\ &\quad + \mathbf{P}_{G^\dagger}(\tau > t_k|\tau \geq t_k)\mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau > t_k\Big) \\ &> \mathbf{P}_{G^\dagger}(\tau = t_k|\tau \geq t_k)F^{1\prime}\Big(X^\dagger_{t_k}\Big) \\ &\quad + \mathbf{P}_{G^\dagger}(\tau > t_k|\tau \geq t_k)\mathbf{E}_{G^\dagger}\Big(F^{1\prime}\Big(X^\dagger_\tau\Big)\Big|\tau > t_k\Big) \\ &= \mathbf{E}_{G^\dagger}\Big(F^{1\prime}\Big(X^\dagger_\tau\Big)\Big|\tau \geq t_k\Big),\end{aligned}$$

where the weak inequality holds since $G|_{\tau \geq t_k}$ MLR-dominates $G^\dagger|_{\tau \geq t_k}$ and

$$F^{1\prime}(X_{t_k}) \leq \mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau > t_k\Big) \quad \text{since } X \text{ and } F^{1\prime} \text{ are decreasing,}$$

and the strict inequality holds by (3) and strict concavity of F^1 (first term) and the fact that (4) holds at $k+1$ (second term).

Induction step: Assume that (3) and (4) hold at $k+1 \leq K$; we must show that they hold at k. Since (4) holds at $k+1$, we have

$$\begin{aligned} F^{0\prime}(x_{t_k}) &= \mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau \geq t_{k+1}\Big) \\ &> \mathbf{E}_{G^\dagger}\Big(F^{1\prime}\Big(X_\tau^\dagger\Big)\Big|\tau \geq t_{k+1}\Big) = F^{0\prime}\Big(x_{t_k}^\dagger\Big), \end{aligned}$$

so that $x_{t_k} < x_{t_k}^\dagger$ by strict concavity of F^0. Using (S) and the fact that (3) holds at $k+1$ yields

$$\begin{aligned} X_{t_k} &= \Big(1 - e^{-r(t_{k+1}-t_k)}\Big)x_{t_k} + e^{-r(t_{k+1}-t_k)}X_{t_{k+1}} \\ &< \Big(1 - e^{-r(t_{k+1}-t_k)}\Big)x_{t_k}^\dagger + e^{-r(t_{k+1}-t_k)}X_{t_{k+1}}^\dagger = X_{t_k}^\dagger, \end{aligned}$$

showing that (3) holds at k. Since (3) holds at k and (4) holds at $k+1$, the blue argument above implies that (4) holds at k as well. ■

5 Proof of Proposition 1

Enumerate the support of G as $\operatorname{supp}(G) = \{t_k\}_{k=1}^K \subseteq \mathbf{R}_+$, where $K \in \mathbf{N}$ and

$$0 \leq t_1 < \cdots < t_K < \infty.$$

As $F^{0\prime}$ is continuous and strictly decreasing on $[u^\star, u^0]$, it admits a continuous and decreasing inverse $\operatorname{inv} F^{0\prime} : [F^{0\prime}(u^0), F^{0\prime}(u^\star)] \to [u^\star, u^0]$. Extend $\operatorname{inv} F^{0\prime}$ to $\mathbf{R}$ by making it constant on $(-\infty, F^{0\prime}(u^0)]$ and on $[F^{0\prime}(u^\star), \infty)$, so that continuity and monotonicity are preserved.

For $\lambda \in [u^\star, u^0]$, let $x_{t_K}^\lambda \coloneqq X_{t_K}^\lambda \coloneqq \lambda$ and, if $K > 1$, define a sequence

$$\Big\{x_{t_k}^\lambda, X_{t_k}^\lambda\Big\}_{k=1}^{K-1}$$

in $[u^\star, u^0]$ recursively by

$$\begin{aligned} x_{t_k}^\lambda &\coloneqq \operatorname{inv} F^{0\prime}\Big(\mathbf{E}_G\Big(F^{1\prime}\Big(X_\tau^\lambda\Big)\Big|\tau > t_k\Big)\Big) \quad \text{and} \\ X_{t_k}^\lambda &\coloneqq \Big[1 - e^{r(t_k - t_{k+1})}\Big]x_k^\lambda + e^{r(t_k - t_{k+1})}X_{t_{k+1}}^\lambda. \end{aligned}$$

Claim. The sequence $(x^\lambda_{t_k})_{k=1}^K$ is decreasing.

Proof. We prove that the sequence $(x^\lambda_{t_k})_{k=k'}^K$ is decreasing for every $k' \in \{1,\dots,K-1\}$ by backward induction on k'. For the base case $k' = K-1$, we have

$$x^\lambda_{t_{K-1}} = \operatorname{inv} F^{0\prime}\left(F^{1\prime}(\lambda)\right) \geq \lambda = x^\lambda_{t_K},$$

where the inequality holds since $F^{0\prime} \geq F^{1\prime}$ on $[u^\star, u^0] \ni \lambda$.

For the induction step, suppose for $k' \in \{1,\dots,K-2\}$ that $(x^\lambda_{t_k})_{k=k'+1}^K$ is decreasing; we must show that $x_{t_{k'}} \geq x_{t_{k'+1}}$. The induction hypothesis implies that

$$\left(X^\lambda_{t_k}\right)_{k=k'+1}^K$$

is also decreasing, which since $F^{1\prime}$ is a decreasing function implies that

$$\left(F^{1\prime}\left(X^\lambda_{t_k}\right)\right)_{k=k'+1}^K$$

is increasing, so that in particular

$$F^{1\prime}\left(X^1_{t_{k'+1}}\right) \leq \mathbf{E}_G\left(F^{1\prime}\left(X^1_\tau\right)\middle|\tau > t_{k'+1}\right).$$

It follows that

$$\begin{aligned}\mathbf{E}_G\left(F^{1\prime}\left(X^1_\tau\right)\middle|\tau > t_{k'}\right) &= \frac{G(t_{k'+1}) - G(t_{k'})}{1 - G(t_{k'})} F^{1\prime}\left(X^1_{t_{k'+1}}\right) \\ &\quad + \frac{1 - G(t_{k'+1})}{1 - G(t_{k'})} \mathbf{E}_G\left(F^{1\prime}\left(X^1_\tau\right)\middle|\tau > t_{k'+1}\right) \\ &\leq \mathbf{E}_G\left(F^{1\prime}\left(X^1_\tau\right)\middle|\tau > t_{k'+1}\right).\end{aligned}$$

Since $\operatorname{inv} F^{0\prime}$ is decreasing, it follows that $x_{t_{k'}} \geq x_{t_{k'+1}}$. □

Since $\operatorname{inv} F^{0\prime}$ and $F^{1\prime}$ are continuous, $\lambda \mapsto x^\lambda_{t_k}$ and $\lambda \mapsto X^\lambda_{t_k}$ are continuous on $[u^\star, u^0]$ for every $k \in \{1,\dots,K\}$.[4] Thus the map $\psi : [u^\star, u^0] \to \mathbf{R}$ defined by

$$\psi(\lambda) \coloneqq \mathbf{E}_G\left(F^{1\prime}\left(X^\lambda_\tau\right)\right) \quad \text{for each } \lambda \in \left[u^\star, u^0\right]$$

[4]Proceed by strong backward induction on $k \in \{1,\dots,K\}$. Clearly, continuity holds in the base case $k = K$. For the induction step, suppose for $k < K$ that $\lambda \mapsto X^\lambda_{t_{k'}}$ is continuous for all $k' > k$. Then $\lambda \mapsto x^\lambda_{t_k}$ is continuous, and thus so is $\lambda \mapsto X^\lambda_{t_k}$.

is continuous. Since F^0 and F^1 are continuously differentiable and $u^\star > 0$, we have by definition of $u^\star$ that

$$F^{1\prime}(u^\star) = F^{0\prime}(u^\star) \quad \text{and} \quad F^{1\prime}\Big(u^0\Big) \le F^{0\prime}\Big(u^0\Big).$$

Thus if $\lambda \in \{u^\star, u^0\}$, then $x^\lambda_{t_k} = \lambda$ for every $k \in \{1, \dots, K\}$, so that $\psi(\lambda) = F^{1\prime}(\lambda)$.[5] It follows that

$$\psi(u^\star) = F^{1\prime}(u^\star) \ge 0 \ge F^{1\prime}\Big(u^0\Big) = \psi\Big(u^0\Big).$$

Since ψ is continuous, it has a root $\lambda_\star \in \big[u^\star, u^0\big]$ by the intermediate value theorem.

Let $x : \mathbf{R}_+ \to \big[u^\star, u^0\big]$ be given by

$$x_t \coloneqq \begin{cases} u^0 & \text{for } t \in [0, t_1) \\ x^{\lambda_\star}_{t_k} & \text{for } t \in [t_k, t_{k+1}) \text{ where } k \in \{1, \dots, K-1\} \\ x^{\lambda_\star}_{t_K} & \text{for } t \in [t_k, \infty). \end{cases}$$

Then (x, X) is simply optimal for $\big(F^0, F^1, G\big)$, as desired. ■

6 Proof of Lemma 2

For each $x \in \mathcal{X}'$, let $\Phi(x)$ denote the function $t \mapsto \mathbf{1}_{\mathbf{R}_+}(t) r e^{-rt} x_t$, an element of $L^1(\mathbf{R})$. By inspection, the map Φ satisfies

$$\int_{\mathbf{R}} \Big|\Phi(x) - \Phi\Big(x^\dagger\Big)\Big| = \Big\|x - x^\dagger\Big\| \quad \text{for any } x, x^\dagger \in \mathcal{X}',$$

so is a homeomorphism between $\mathcal{X}'$ and a subspace of $L^1(\mathbf{R})$. In particular, $\mathcal{X}'$ is homeomorphic to the set $\mathcal{F} \subseteq L^1(\mathbf{R})$ of $f : \mathbf{R} \to \mathbf{R}$ such that (up to a.e. equivalence) $f(t) = \mathbf{1}_{\mathbf{R}_+}(t) r e^{-rt} x_t$ for some decreasing $x : \mathbf{R}_+ \to \big[u^\star, u^0\big]$.

Since $\mathcal{F}$ is closed in $L^1(\mathbf{R})$,[6] we need only show that it is relatively compact. To this end, we use the Fréchet–Kolmogorov theorem,[7] which provides that a set $\mathcal{F} \subseteq L^1(\mathbf{R})$ is relatively compact if it satisfies the following:

(a) $\mathcal{F}$ is bounded,

[5] This follows easily by backward induction on $k \in \{1, \dots, K\}$.

[6] Consider a sequence $(f^n)_{n \in \mathbf{N}}$ in $\mathcal{F}$ converging to $f \in \mathcal{F}$. By modifying each f^n on a set of measure 0, we may assume that $f^n(t) = \mathbf{1}_{\mathbf{R}_+}(t) e^{-rt} x^n_t$ for some decreasing $x^n : \mathbf{R}_+ \to \big[0, u^0\big]$. Since $f^n \to f$ a.e., it follows that f belongs to $\mathcal{F}$.

[7] See e.g. Corollary 4.27 in Brezis (2011, p. 113).

(b) for any $\varepsilon > 0$, there is an $R > 0$ such that

$$\int_{(-\infty,-R)\cup(R,\infty)} |f| < \varepsilon \quad \text{for every } f \in \mathcal{F}, \text{ and}$$

(c) for every $\varepsilon > 0$, there is a $\rho > 0$ such that

$$\int_{\mathbf{R}} |f(t+s) - f(t)| \mathrm{d}t < \varepsilon \quad \text{for every } f \in \mathcal{F} \text{ and } s \in \mathbf{R} \text{ with } |s| < \rho.$$

Clearly $\mathcal{F}$ satisfies (a) and (b). To prove that (c) holds, note that for any decreasing $x : \mathbf{R}_+ \to [u^\star, u^0]$ and any $\delta > 0$, we have $\|x^\delta - x\| < \delta$ for some decreasing step function $x^\delta : \mathbf{R}_+ \to [u^\star, u^0]$:

$$x_t^\delta = u^\star + \sum_{l=1}^{L} \alpha_l \mathbf{1}_{[0,t_l]}(t)$$

for some $0 \leq t_1 < \cdots < t_L$ (with $L \in \mathbf{N}$) and $\alpha_1, \ldots, \alpha_L \geq 0$ satisfying

$$u^\star + \sum_{l=1}^{L} \alpha_l \leq u^0.$$

For any $s \in \mathbf{R}$, we have

$$\int_{\mathbf{R}} \left|\Phi\left(x^\delta\right)(t+s) - \Phi(x)(t+s)\right| \mathrm{d}t = \int_{\mathbf{R}} \left|\Phi\left(x^\delta\right) - \Phi(x)\right| = \left\|x^\delta - x\right\| < \delta$$

and

$$\begin{aligned}
&\int_{\mathbf{R}} \left|\Phi\left(x^\delta\right)(t+s) - \Phi\left(x^\delta\right)(t)\right| \mathrm{d}t \\
&\qquad = \int_{\mathbf{R}} \left|\mathbf{1}_{[-s,\infty)}(t) r e^{-r(t+s)} x_{t+s}^\delta - \mathbf{1}_{\mathbf{R}_+}(t) r e^{-rt} x_t^\delta\right| \mathrm{d}t \\
&\qquad \leq r u^0 |s| + \int_0^\infty r e^{-rt} \left|x_{t+s}^\delta - x_t^\delta\right| \mathrm{d}t \\
&\qquad \leq r u^0 |s| + r \int_0^\infty \left|x_{t+s}^\delta - x_t^\delta\right| \mathrm{d}t \\
&\qquad = r u^0 |s| + r|s| \sum_{l=1}^{L} \alpha_l \\
&\qquad \leq r|s| \left(2u^0 - u^\star\right).
\end{aligned}$$

Now fix any $\varepsilon > 0$, and let

$$\rho \coloneqq \frac{1}{r(2u^0 - u^\star)}\frac{\varepsilon}{3} \quad \text{and} \quad \delta \coloneqq \frac{\varepsilon}{3}.$$

Then for any $f \in \mathcal{F}$ and $s \in \mathbf{R}$ with $|s| < \rho$, writing $x \coloneqq \Phi^{-1}(f)$ and $f^\delta \coloneqq \Phi(x^\delta)$, we have

$$\begin{aligned}\int_{\mathbf{R}} |f(t+s) - f(t)| \mathrm{d}t &\leq \int_{\mathbf{R}} \left| f(t) - f^\delta(t) \right| \mathrm{d}t + \int_{\mathbf{R}} \left| f(t+s) - f^\delta(t+s) \right| \mathrm{d}t \\ &\quad + \int_{\mathbf{R}} \left| f^\delta(t+s) - f^\delta(t) \right| \mathrm{d}t \\ &< \delta + \delta + r|s|\left(2u^0 - u^\star\right) \leq \varepsilon.\end{aligned}$$

Thus (c) holds. ■

7 Proof of Lemma 3

Observation 2. Suppose for a mechanism (x, X) that there are bounded and measurable $\phi^0, \phi^1 : \mathbf{R}_+ \to \mathbf{R}$ such that $\phi^0(t)$ ($\phi^1(t)$) is a supergradient of F^0 at x_t (of F^1 at X_t) for every $t \in \mathbf{R}_+$,

$$\mathbf{E}_G\left(\phi^1(\tau)\right) = 0, \tag{5}$$

and

$$\phi^0(t) = \mathbf{E}_G\left(\phi^1(\tau) \middle| \tau > t\right) \quad \text{for a.e. } t \in \mathbf{R}_+ \text{ at which } G(t) < 1. \tag{6}$$

Then (x, X) satisfies the Euler equation (appendix E, p. 41 of the paper).

Proof. Let A be the null set of times $t \in \mathbf{R}_+$ at which $G(t) < 1$ and (6) fails. Note that $t \mapsto \mathbf{E}_G(\phi^1(\tau)|\tau > t)$ is right-continuous. Define $\widehat{\phi}^0 : \mathbf{R}_+ \to \mathbf{R}_+$ by

$$\widehat{\phi}^0(t) = \begin{cases} \lim_{s \downarrow t} \mathbf{E}_G(\phi^1(\tau)|\tau > s) & \text{if } t \in A \\ \phi^0(t) & \text{if } t \notin A. \end{cases}$$

Then $(\widehat{\phi}^0, \phi^1)$ satisfy (6) at every $t \in \mathbf{R}_+$ such that $G(t) < 1$. Define $\widehat{x} : \mathbf{R}_+ \to \mathbf{R}_+$ by

$$\widehat{x}_t = \begin{cases} \inf_{t' > t} \sup\{x_s : s \in (t, t') \setminus A\} & \text{for } t \in A \\ x_t & \text{for } t \notin A. \end{cases}$$

Clearly $\widehat{x} = x$ a.e., and $\widehat{\phi}^0(t)$ is a supergradient of F^0 at $\widehat{x}_t = x_t$ for $t \notin A$. It remains only to show that $\widehat{\phi}^0(t)$ is a supergradient of F^0 at $\widehat{x}_t$ for each $t \in A$.

So fix a $t \in A$. By definition of $\widehat{x}_t$, there is a decreasing sequence $(t_n)_{n\in\mathbf{N}}$ in $\mathbf{R}_+ \setminus A$ converging to t along which x_{t_n} converges to $\widehat{x}_t$. Then

$$\widehat{\phi}^0(t) = \lim_{n\to\infty} \mathbf{E}_G\Big(\phi^1(\tau)\Big|\tau > t_n\Big) = \lim_{n\to\infty} \phi^0(t_n)$$

where the first equality holds since $t_n \downarrow t$, and the second since $(t_n)_{n\in\mathbf{N}}$ lives in $\mathbf{R}_+ \setminus A$. Since $\phi^0(t_n)$ is a supergradient of F^0 at x_{t_n} for every $n \in \mathbf{N}$ and $x_{t_n} \to \widehat{x}_t$, $\lim_{n\to\infty} \phi^0(t_n) = \widehat{\phi}^0(t)$ is a supergradient of F^0 at $\widehat{x}_t$. ■

Proof of Lemma 3. Define $\phi^0, \phi^1 : \mathbf{R}_+ \to \mathbf{R}$ by

$$\phi^0(t) := F^{0\prime}(x_t) \quad \text{and} \quad \phi^1(t) := F^{1\prime}(X_t) \quad \text{for each } t \in \mathbf{R}_+.$$

By Observation 2, it suffices to show that (5) and (6) are satisfied.

For the former, note that (by definition of weak convergence)

$$\mathbf{E}_{G_n}\Big(F^{1\prime}(X_\tau)\Big) \to \mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big). \tag{7}$$

since X and $F^{1\prime}$ are continuous and the latter is bounded. By hypothesis, $F^{1\prime}$ is L-Lipschitz over $[u^\star, u^0]$ for some $L > 0$. So for any $T \in \mathbf{R}_+$, we have

$$\begin{aligned}
&\Big|\mathbf{E}_{G_n}\Big(F^{1\prime}(X^n_\tau) - F^{1\prime}(X_\tau)\Big)\Big| \\
&\qquad\qquad \leq L\mathbf{E}_{G_n}(|X^n_\tau - X_\tau|) \\
&\qquad\qquad \leq L\mathbf{E}_{G_n}\Big(|X^n_\tau - X_\tau| \;\Big|\; \tau \leq T\Big) + [1 - G_n(T)]L\Big(u^0 - u^\star\Big) \\
&\qquad\qquad \leq L\mathbf{E}_{G_n}\left(e^{r\tau}\int_\tau^\infty re^{-rt}|x^n_t - x_t|\mathrm{d}t \,\middle|\, \tau \leq T\right) \\
&\qquad\qquad\quad + L[1 - G_n(T)]\Big(u^0 - u^\star\Big) \\
&\qquad\qquad \leq Le^{rT}\|x^n - x\| + L[1 - G_n(T)]\Big(u^0 - u^\star\Big).
\end{aligned}$$

Since $T \in \mathbf{R}_+$ was arbitrary and $\|x^n - x\| \to 0$, it follows that

$$\Big|\mathbf{E}_{G_n}\Big(F^{1\prime}(X^n_\tau) - F^{1\prime}(X_\tau)\Big)\Big| \to 0.^8 \tag{8}$$

[8] Fix any $\varepsilon > 0$; we seek an $N \in \mathbf{N}$ such that $|\mathbf{E}_{G_n}[F^{1\prime}(X^n_\tau) - F^{1\prime}(X_\tau)]| < \varepsilon$ for all $n \geq N$. To this end, choose a $T \in \mathbf{R}_+$ so that $[1 - G(T)]L(u^0 - u^\star) < \varepsilon/3$ and $G_n(T) \to G(T)$, and an $N \in \mathbf{N}$ such that for all $n \geq N$ it holds that $|G_n(T) - G(T)|L(u^0 - u^\star) < \varepsilon/3$ and that $Le^{rT}\|x^n - x\| < \varepsilon/3$.

Together, (7) and (8) imply that

$$\mathbf{E}_{G_n}\Big(F^{1\prime}(X^n_\tau)\Big) \to \mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big).$$

Thus since x^n satisfies (5) for each $n \in \mathbf{N}$, so does x.

It remains to derive (6). Note that G has at most countably many atoms. Since $x^n \to x$ a.e., it suffices to prove that (6) holds for all $t \in \mathbf{R}_+$ at which $G(t) < 1$, G has no atom at t, and $x^n_t \to x_t$. Fix such a $t \in \mathbf{R}_+$. Since G does not have an atom at t, $\big(G_n|_{(t,\infty)}\big)_{n\in\mathbf{N}}$ converges weakly to $G|_{(t,\infty)}$. A straightforward variation on the above argument then yields

$$\mathbf{E}_{G_n}\Big(F^{1\prime}(X^n_\tau)\Big|\tau > t\Big) \to \mathbf{E}_G\Big(F^{1\prime}(X_\tau)\Big|\tau > t\Big).$$

Furthermore, $F^{0\prime}(x^n_t) \to F^{0\prime}(x_t)$ since $F^{0\prime}$ is continuous and $x^n_t \to x_t$. Thus since x^n for each $n \in \mathbf{N}$ satisfies (6) at t, so does x. ■

8 Proof of Proposition 3

For $j \in \{0,1\}$, write $F^{j-}(u)$ ($F^{j+}(u)$) for the left-hand (right-hand) derivative of F^j at $u >(\geq)\ 0$. It is clear that there exists a sequence $\big(F^0_n, F^1_n\big)_{n\in\mathbf{N}}$ such that all of the following hold:[9]

(a) for every $n \in \mathbf{N}$, $F^0_n, F^1_n : \mathbf{R}_+ \to \mathbf{R}$ are simple,

(b) for both $j \in \{0,1\}$ and every sequence $u_n \to u$, any subsequential limit of the sequence $\big(F^{j\prime}_n(u_n)\big)_{n\in\mathbf{N}}$ is a supergradient of F^j at u,[10] and if

[9]Here's an example. Fix a $k > 0$, and define $f^0_n, f^1_n : (0,\infty) \to \mathbf{R}$ for each $n \in \mathbf{N}$ by

$$f^0_n(u) \coloneqq F^{0\prime-}(u) - \frac{1}{kn}F^{0\prime+}(0)u \quad\text{and}\quad f^1_n(u) \coloneqq \begin{cases} k & \text{for } u \leq 2/n \\ F^{1\prime-}(u) & \text{for } u > 2/n. \end{cases}$$

For $j \in \{0,1\}$, let $\varphi_n : \mathbf{R}_+ \to \mathbf{R}$ be given by

$$\varphi^j_n(u) \coloneqq \frac{1}{1/n + \min\{u, 1/n\}} \int_{\max\{u-1/n,0\}}^{u+1/n} f^j_n,$$

and define $F^j_n(u) \coloneqq \int_0^u \varphi^j_n$ for each $u \in \mathbf{R}_+$. This sequence has the requisite properties when $k > 0$ is chosen large enough. (a) and (c) are straightforward; we shall verify (b). Fix a sequence $u_n \to u \in \mathbf{R}_+$, and note that for all $n \in \mathbf{N}$,

$$F^{j\prime}_n(u_n) = \varphi^j_n(u_n) \in \big[f^j_n(\max\{u_n - 1/n, 0\}), f^j_n(u_n + 1/n)\big]$$

since f^j_n is decreasing. Furthermore,

$$\liminf_{n\to\infty} f^j_n(\max\{u_n - 1/n, 0\}) \geq F^{j+}(u) \quad\text{and}\quad \limsup_{n\to\infty} f^j_n \leq \begin{cases} k & \text{if } u = 0 \\ F^{0-}(u) & \text{if } u > 0. \end{cases}$$

[10]Note that $-\infty$ and ∞ can be subsequential limits.

$u = 0$ then this set of subsequential limits is bounded above, and

(c) the sequence $(F_n^{1\prime})_{n\in\mathbf{N}}$ is uniformly bounded on $[0, u^0]$.

Fix a mechanism x and G, and suppose that there exists $(x^n)_{n\in\mathbf{N}} \subseteq \mathcal{X}'$ such that $x^n \to x$ a.e. and, for all $n \in \mathbf{N}$, (x^n, X^n) satisfies the Euler equation for F_n^0, F_n^1 and G. It remains to prove that (x, X) satisfies the Euler equation.

Claim. There exists a subsequence $(x^{n,\star})_{n\in\mathbf{N}}$ of $(x^n)_{n\in\mathbf{N}}$ such that

(i) $\lim_{n\to\infty} F_n^{0\prime}(x_t^{n,\star})$ exists and is a supergradient of F^0 at x_t

for (Lebesgue-)almost every $t \in \mathbf{R}_+$, and

(ii) $\lim_{n\to\infty} F_n^{1\prime}(X_t^{n,\star})$ exists and is a supergradient of F^1 at X_t

for G-almost every $t \in \mathbf{R}_+$.

Proof. We shall construct a subsequence $(x^{n,\star})_{n\in\mathbf{N}}$ of $(x^n)_{n\in\mathbf{N}}$ such that (i) holds for a.e. and G-a.e. $t \in \mathbf{R}_+$ at which $x_t^n \to x_t$. The same argument may then be applied to obtain a sub-subsequence such that (ii) holds for a.e. and G-a.e. $t \in \mathbf{R}_+$ at which $X_t^n \to X_t$. This is sufficient because $x^n \to x$ a.e. and $X^n \to X$ pointwise by $\|x^n - x\| \to 0$, so that the sub-subsequence satisfies (i) for a.e. (and G-a.e.) $t \in \mathbf{R}_+$ and satisfies (ii) for (a.e. and) G-a.e. $t \in \mathbf{R}_+$.

By property (b), we have $F_n^{0\prime}(x_t^n) \to F^{0\prime}(x_t)$ for a.e. $t \in \mathbf{R}_+$ at which $x_t^n \to x_t$ and F^0 is differentiable at x_t. Let U be the set of $u > 0$ at which F^0 is not differentiable. Since F^0 is concave, $U \cup \{0\}$ is countable, so may be enumerated as $U = \{u_k\}_{k=1}^K$ where $K \in \mathbf{N} \cup \{\infty\}$.

We claim that for any $u \in U \cup \{0\}$, every subsequence $(x^{n,\natural})_{n\in\mathbf{N}}$ of $(x^n)_{n\in\mathbf{N}}$ admits a subsequence $(x^{n,\star})_{n\in\mathbf{N}}$ such that property (i) holds for a.e. and G-a.e. $t \in \mathbf{R}_+$ at which $x_t^n \to x_t = u$. This is sufficient because it implies the existence of subsequences

$$(x^n)_{n\in\mathbf{N}} \supseteq \left(x^{n,1}\right)_{n\in\mathbf{N}} \supseteq \left(x^{n,2}\right)_{n\in\mathbf{N}} \supseteq \cdots$$

such that for each $k \in \mathbf{N}$, the subsequence $(x_t^{n,k})_{n\in\mathbf{N}}$ satisfies property (i) at a.e. and G-a.e. $t \in \mathbf{R}_+$ at which

$$x_t^n \to x_t \in \{u_l\}_{l=1}^k,$$

so that $x^{n,\star} := x^{n,n}$ is a subsequence of $(x^n)_{n\in\mathbf{N}}$ for which (i) holds for a.e. and G-a.e. $t \in \mathbf{R}_+$ at which $x_t^n \to x_t$.

To prove this claim, fix a $u \in U \cup \{0\}$ and a subsequence $(x^{n,\natural})_{n\in\mathbf{N}}$ of $(x^n)_{n\in\mathbf{N}}$. Define

$$I \coloneqq \left\{ t \in \mathbf{R}_+ : \lim_{n\to\infty} x_t^{n,\natural} = u \right\},$$

and let $\mathcal{S}$ be the set of subsequences of $(x^{n,\natural})_{n\in\mathbf{N}}$. For each $s = (y^n)_{n\in\mathbf{N}} \in \mathcal{S}$ and $t \in I$, write $\psi^s(t) \subseteq [-\infty, \infty]$ for the set of all subsequential limits of the real sequence $(F_n^{0\prime}(y_t^n))_{n\in\mathbf{N}}$. For every $s \in \mathcal{S}$, ψ^s is a correspondence $I \rightrightarrows [-\infty, \infty]$. We seek a subsequence $s_\star \in \mathcal{S}$ and a set $A \subseteq I$ such that ψ^s is singleton-valued on A and $I \setminus A$ is Lebesgue- and G-null.

Fix any $s = (y^n)_{n\in\mathbf{N}} \in \mathcal{S}$. Clearly ψ^s is non-empty- and closed-valued. In fact, ψ^s takes compact values $\subseteq \mathbf{R}$: for any $t \in I$, we have $y^n \to u$ by definition of I, so that $\psi^s(t)$ is bounded by property (b). Furthermore, the maps

$$\min \psi^s, \max \psi^s : I \to \mathbf{R}_+$$

are increasing, because $t \mapsto F_n^{0\prime}(y_t^n)$ is increasing since $F_n^{0\prime}$ and y^n are decreasing.

Let B be the set of atoms of G, and define

$$C \coloneqq (\{\inf I\} \cup B \cup \mathbf{Q}) \cap I.$$

This set is at most countable since B is, so may be enumerated as $C = \{t_k\}_{k\in\mathbf{N}}$. Write $s_0 \coloneqq (x^{n,\natural})_{n\in\mathbf{N}}$, and choose subsequences $s_1, s_2, \ldots$ so that

$$s_k \subseteq s_{k-1} \quad \text{and} \quad \psi^{s_k}(t_k) = \{\max \psi^{s_{k-1}}(t_k)\} \quad \text{for each } k \in \mathbf{N}.$$

Write $s_k = (x^{n,k})_{n\in\mathbf{N}}$ for the kth subsequence, and define $s_\star \coloneqq (x^{n,n})_{n\in\mathbf{N}}$.

Let

$$A \coloneqq \{t \in I : t \in C \text{ or } \max \psi^{s_\star} \text{ is continuous at } t\}.$$

We must show that $I \setminus A$ is Lebesgue- and G-null and that $\psi^{s_\star}$ is singleton-valued on A. The former is easy: since $\max \psi^s$ is increasing, $I \setminus A$ is at most countable, hence Lebesgue-null. Because $B \cap I \subseteq A$, it follows that $I \setminus A$ is also G-null.

For the latter, fix a $t \in A$. If $t \in C = \{t_k\}_{k\in\mathbf{N}}$, then $\psi^s(t)$ is a singleton since

$$\psi^{s_\star}(t_k) = \{\max \psi^{s_{k-1}}(t_k)\} \quad \text{for every } k \in \mathbf{N}.$$

Suppose instead that $t \in A \setminus C$, so that $\max \psi^{s_\star}$ is continuous at t. Assume toward a contradiction that $z < z'$ both belong to $\psi^{s_\star}(t)$. The set I is convex since $x^{n,\natural}$ is decreasing for every $n \in \mathbf{N}$, and C is clearly dense in I.

Furthermore, t strictly exceeds $\inf I$ since the latter belongs to C. Thus there is a strictly increasing sequence $(t'_k)_{k\in\mathbf{N}}$ in C converging to t, along which

$$\lim_{k\to\infty} \max \psi^{s_\star}(t'_k) = \max \psi^{s_\star}(t) \geq z' > z$$

by continuity at t. Thus there is a $k \in \mathbf{N}$ at which $\max \psi^{s_\star}(t'_k) > z$. Because $\psi^{s_\star}(t'_k)$ is a singleton (since t'_k belongs to C), we then have

$$\min \psi^{s_\star}(t'_k) = \max \psi^{s_\star}(t'_k) > z \geq \min \psi^{s_\star}(t),$$

which since $t'_k < t$ contradicts the fact that $\min \psi^{s_\star}$ is increasing. □

Fix a subsequence $(x^{n,\star})_{n\in\mathbf{N}}$ that satisfies the conditions of the claim, and define $\phi^0, \phi^1 : \mathbf{R}_+ \to \mathbf{R}$ by

$$\phi^0(t) \coloneqq \begin{cases} \lim_{n\to\infty} F_n^{0\prime}(x_t^{n,\star}) & \text{if the limit exists} \\ F^{0+}(x_t) & \text{otherwise} \end{cases}$$

$$\phi^1(t) \coloneqq \begin{cases} \lim_{n\to\infty} F_n^{1\prime}(X_t^{n,\star}) & \text{if the limit exists} \\ F^{1+}(X_t) & \text{otherwise.} \end{cases}$$

Then ϕ^0 and ϕ^1 are bounded and measurable with $\phi^0(t)$ ($\phi^1(t)$) a supergradient of F^0 at x_t (of F^1 at X_t). By Observation 2, it suffices to show that (5) and (6) on page 11 hold.

For the former, we have

$$\mathbf{E}_G\Big(F_n^{1\prime}(X_\tau^{n,\star})\Big) = 0 \quad \text{for each } n \in \mathbf{N}$$

by construction of $x^{n,\star}$, and

$$F_n^{1\prime}(X_t^{n,\star}) \to \phi^1(t) \quad \text{for } G\text{-a.e. } t \in \mathbf{R}_+$$

by property (ii). Since $(F_n^{1\prime})_{n\in\mathbf{N}}$ is uniformly bounded on $[0, u^0]$ by property (c), and $X^{n,\star}$ takes values in $[0, u^0]$, the dominated convergence theorem yields

$$\mathbf{E}_G\Big(\phi^1(\tau)\Big) = \lim_{n\to\infty} \mathbf{E}_G\Big(F_n^{1\prime}(X_t^{n,\star})\Big) = 0,$$

which is (5).

For the latter, we have for each $n \in \mathbf{N}$ that

$$F_n^{0\prime}(x_t^{n,\star}) = \mathbf{E}_G\Big(F_n^{1\prime}(X_t^{n,\star})\Big|\tau > t\Big) \quad \text{for a.e. } t \in \mathbf{R}_+ \text{ at which } G(t) < 1$$

by construction of $x^{n,\star}$. The argument in the preceding paragraph implies that the right-hand side converges to $\mathbf{E}_G(\phi^1(\tau)|\tau > t)$ for every $t \in \mathbf{R}_+$ at which $G(t) < 1$. The left-hand side converges to $\phi^0(t)$ for a.e. $t \in \mathbf{R}_+$ by $x^{n,\star} \to x$ a.e. and property (i). Thus (6) holds. ■

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