Agenda-manipulation in ranking

Gregorio Curello
University of Bonn

Ludvig Sinander
Northwestern University

22 September 2020

paper: arXiv.org/abs/2001.11341

Ranking by committee

A committee must rank a set of alternatives.

Hiring:

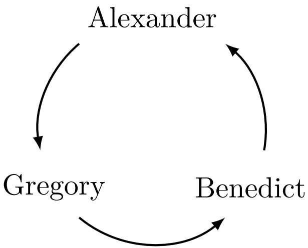
- alternatives are candidates for a job
- uncertainty about who will accept
- hiring committee decides to whom to offer the job, to whom next if the first candidate declined, etc.

Party lists:

- alternatives are a political party's parliamentary candidates
- party's leadership committee ranks them ('party list')
- the K highest-ranked candidates get parliamentary seats, where K is (uncertain) # seats the party wins in an election

Interaction

The majority will may contain (Condorcet) cycles:



The committee's *chair* chooses the order of pairwise votes.

Transitivity is imposed.

Preferences

The chair has a preference $\succ$ over alternatives.

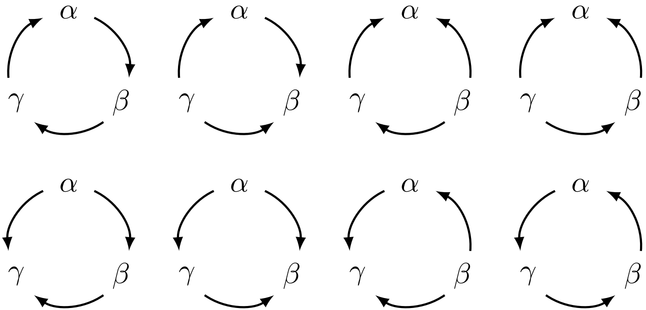
Ranking R is *more aligned with* $\succ$ than R'
iff whenever $x \succ y$ and $x \; R' \; y$, also $x \; R \; y$.

The chair prefers rankings that are more aligned with $\succ$.

Hiring: a more aligned ranking is exactly one that hires a $\succ$-better candidate at every realisation of uncertainty.

Unknown majority will

The chair does not know the majority will, W.



Regret-free strategies

A ranking is W-*unimprovable* iff no other ranking is both

(i) reachable under W and

(ii) more aligned with $\succ$.

With perfect knowledge of W,
W-unimprovability is the strongest optimality concept.

A *regret-free* strategy
reaches a W-unimprovable ranking under every W.

Results

We introduce a strategy called *insertion sort.*

Theorem 1. Insertion sort is regret-free.

What (other) strategies are regret-free?

Characterisation of *outcomes:*
Theorem 2. A strategy is regret-free iff it is *efficient.*

Characterisation of *behaviour:*
Theorem 3. A strategy is regret-free iff it avoids two errors.

What's special about insertion sort?
Theorem 4. IS is characterised by a lexicographic property.

Related literature

- **agenda-manipulation:** Farquharson (1969), Black (1958), Miller (1977), Banks (1985)

  ... with incomplete info: Ordeshook and Palfrey (1988), recent work by Benny Moldovanu & co-authors

- **social choice:** Zermelo (1929), Wei (1952), Kendall (1955)

  ... Copeland's method: Copeland (1951), Rubinstein (1980)

  ... Kemeny–Slater method: Kemeny (1959), Slater (1961), Young and Levenglick (1978), Young (1986, 1988)

  ... fair-bets method: Daniels (1969), Moon and Pullman (1970), Slutzki and Volij (2005)

(references: slide 47)

Environment

Finite set $\mathcal{X}$ of alternatives.

Definition.
Ranking: an irreflexive, total & transitive relation on $\mathcal{X}$.

For a ranking R, $x\ R\ y$ reads 'x is ranked above y'.

Characters:

– committee of voters $i \in \{1, \ldots, I\}$, where I is odd
– the committee's chair

Interaction

Write R_t for what has been decided by the end of period t.
(A *proto-ranking:* an irreflexive, ~~total~~ & transitive relation on $\mathcal{X}$.)

Initially, nothing is decided: $R_0 = \varnothing$.

In each period t, unless R_{t-1} is already total,

- chair offers vote on an unranked (by R_{t-1}) pair $x, y \in \mathcal{X}$
- each voter $i \in \{1, \ldots, I\}$ votes for either x or y
- winner is ranked above loser, and transitivity is imposed:

$$R_t = \text{transitive closure of} \begin{cases} R_{t-1} \cup \{(x,y)\} & \text{if } x \text{ won} \\ R_{t-1} \cup \{(y,x)\} & \text{if } y \text{ won.} \end{cases}$$

Why this protocol?

Our *transitive protocol* denies the chair arbitrary power:

– *committee sovereignty:*
if x beats y in a vote, then x is ranked above y.

– *democratic legitimacy:*
enough votes must be offered that
every pair is linked by a chain of majorities.

Any protocol that denies the chair arbitrary power
is exactly the transitive protocol
with restrictions on which unranked pairs the chair can offer.

The majority will

Fix how each voter votes on each pair.

Write $x \; W \; y$ if x beats y in a pairwise vote.

The majority will W is all that matters to the chair.

Fact: all and only total and asymmetric relations can be majority wills.

History-invariant voting

By using the majority will, we implicitly assume (approximately) history-invariant voting.

– reasonable if voting is non-strategic or ‘expressive’

– not unreasonable if voting is strategic

Strategies

A *history* specifies a sequence of pairs offered for a vote and a winner of each vote.

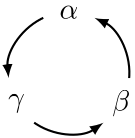
A *strategy* specifies what pair to offer after each history.

The *outcome* of a strategy under W is the ranking that results.

Feasibility

Definition.
Given a majority will W, a ranking is W*-feasible* iff it is the outcome under W of some strategy.

Example.



W-feasible rankings:
$\beta\ R\ \alpha\ R\ \gamma$, $\quad \alpha\ R'\ \gamma\ R'\ \beta \quad$ and $\quad \gamma\ R''\ \beta\ R''\ \alpha$.

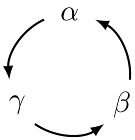
Preferences

The chair has a preference $\succ$ over alternatives.

Ranking R is *more aligned with* $\succ$ than R'
iff whenever $x \succ y$ and $x \; R' \; y$, also $x \; R \; y$.

The chair prefers rankings that are more aligned with $\succ$.

Example



and $\alpha \succ \beta \succ \gamma$

W-feasible rankings:
$\beta \; R \; \alpha \; R \; \gamma, \quad \alpha \; R' \; \gamma \; R' \; \beta \quad \text{and} \quad \gamma \; R'' \; \beta \; R'' \; \alpha.$

R and R' are more aligned with $\succ$ than R''
and are incomparable to each other.

$\implies$ R and R' cannot be W-feasibly improved upon.

Regret-free strategies

Definition.
Given a majority will W, a ranking is W*-unimprovable* iff there is no other W-feasible ranking that is more aligned with $\succ$.

With perfect knowledge of W,
W-unimprovability is the strongest optimality concept.

Definition.
A strategy is *regret-free* iff for any majority will W,
its outcome under W is W-unimprovable.

Roadmap

Efficiency

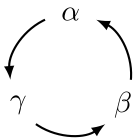
Insertion sort is regret-free (Theorem 1)

Characterisation of regret-free strategies (Theorems 2 & 3)

Efficiency

Definition.
Given a majority will W, a ranking R is W-*efficient* iff
for any pair $x, y \in \mathcal{X}$ with $x \succ y$ and $x\ W\ y$, we have $x\ R\ y$.

Example.



and $\quad \alpha \succ \beta \succ \gamma$

W-efficient rankings: $\quad \succ$ itself, $\quad \beta\ R\ \alpha\ R\ \gamma \quad$ and $\quad \alpha\ R'\ \gamma\ R'\ \beta$.

Definition.
A strategy is *efficient* iff for any majority will W,
its outcome under W is W-efficient.

W-efficiency implies W-unimprovability

Lemma 1.
For any majority will W,
a W-efficient ranking is W-unimprovable.

Corollary.
Any efficient strategy is regret-free.

Proof of Lemma 1

Fix a W, a W-efficient R, and a W-feasible $R' \neq R$.
Suppose toward a contradiction that R' is MAW $\succ$ than R.

Since $R' \neq R$, $\exists$ alternatives x, y such that $x \; R' \; y$ and $y \; R \; x$.
Enumerate the alternatives that R' ranks between x and y as

$$x = z_1 \; R' \; z_2 \; R' \cdots R' \; z_N = y.$$

Since R' is W-feasible, we must have $z_1 \; W \; z_2 \; W \cdots W \; z_N$.

There has to be $n < N$ at which $z_{n+1} \; R \; z_n$,
else we'd have $x \; R \; y$ by transitivity of R.

It must be that $z_{n+1} \succ z_n$,
else we'd have $z_n \; R \; z_{n+1}$ by $z_n \; W \; z_{n+1}$ and W-efficiency of R.

So (z_n, z_{n+1}) is ranked 'right' by R and 'wrong' by R'
. . . which is absurd since R' is MAW $\succ$ than R. ■

Insertion sort

Label the alternatives $\mathcal{X} \equiv \{1, \ldots, n\}$ so that $1 \succ \cdots \succ n$.

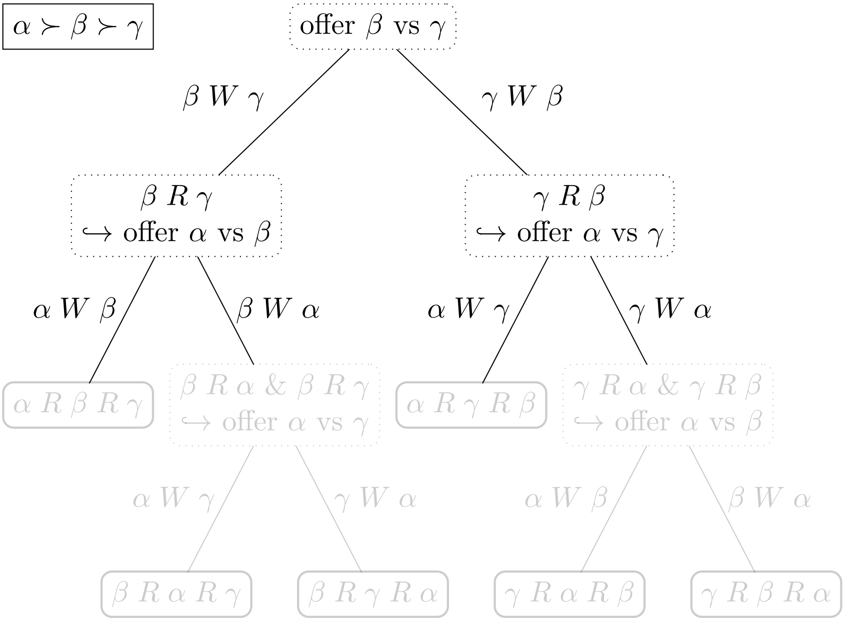
Insertion sort strategy: for each $k \in \{n-1, \ldots, 1\}$,

– totally rank $\{k+1, \ldots, n\}$

(write $x_{k+1} \; R \cdots R \; x_n$, where $\{x_{k+1}, \ldots, x_n\} \equiv \{k+1, \ldots, n\}$)

– 'insert' k into $\{k+1, \ldots, n\}$:

pit k against the highest-ranked (x_{k+1});
then (if k lost) pit k against the 2^{nd}-highest-ranked (x_{k+2});
$\ldots$



$\alpha \succ \beta \succ \gamma$
offer β vs γ
$\beta\ W\ \gamma$
$\gamma\ W\ \beta$
$\beta\ R\ \gamma$ $\hookrightarrow$ offer α vs β
$\gamma\ R\ \beta$ $\hookrightarrow$ offer α vs γ
$\alpha\ W\ \beta$
$\beta\ W\ \alpha$
$\alpha\ W\ \gamma$
$\gamma\ W\ \alpha$
$\alpha\ R\ \beta\ R\ \gamma$
$\beta\ R\ \alpha$ & $\beta\ R\ \gamma$ $\hookrightarrow$ offer α vs γ
$\alpha\ R\ \gamma\ R\ \beta$
$\gamma\ R\ \alpha$ & $\gamma\ R\ \beta$ $\hookrightarrow$ offer α vs β
$\alpha\ W\ \gamma$
$\gamma\ W\ \alpha$
$\alpha\ W\ \beta$
$\beta\ W\ \alpha$
$\beta\ R\ \alpha\ R\ \gamma$
$\beta\ R\ \gamma\ R\ \alpha$
$\gamma\ R\ \alpha\ R\ \beta$
$\gamma\ R\ \beta\ R\ \alpha$

Insertion sort is regret-free

Theorem 1.
The insertion-sort strategy is efficient, hence regret-free.

Proof of Theorem 1

Fix a W, and let R be the outcome of IS under W.
Fix x, y with $x \succ y$ and $x \; W \; y$; we must show that $x \; R \; y$.

Enumerate all alternatives $\succ$-worse than x as $\quad z_1 \; R \cdots R \; z_K$.
Note that $z_k = y$ for some $k \leq K$.

By definition of IS,
x is pitted against $z_1, z_2, \ldots$ in turn until it wins a vote.

- if x loses against $z_1, \ldots, z_{k-1}$,
  then it is pitted against $z_k = y$ and wins (since $x \; W \; y$)
  $\implies x \; R \; y$.

- if x wins against z_ℓ for $\ell < k$,
  then $x \; R \; z_\ell \; R \cdots R \; z_k = y$
  $\implies x \; R \; y$ (by transitivity of R). ■

What (other) strategies are regret-free?

We've shown that regret-free strategies exist.

What are their characteristics?

Characterisation of outcomes

Recall that W-efficiency $\Longrightarrow$ W-unimprovability (Lemma 1).

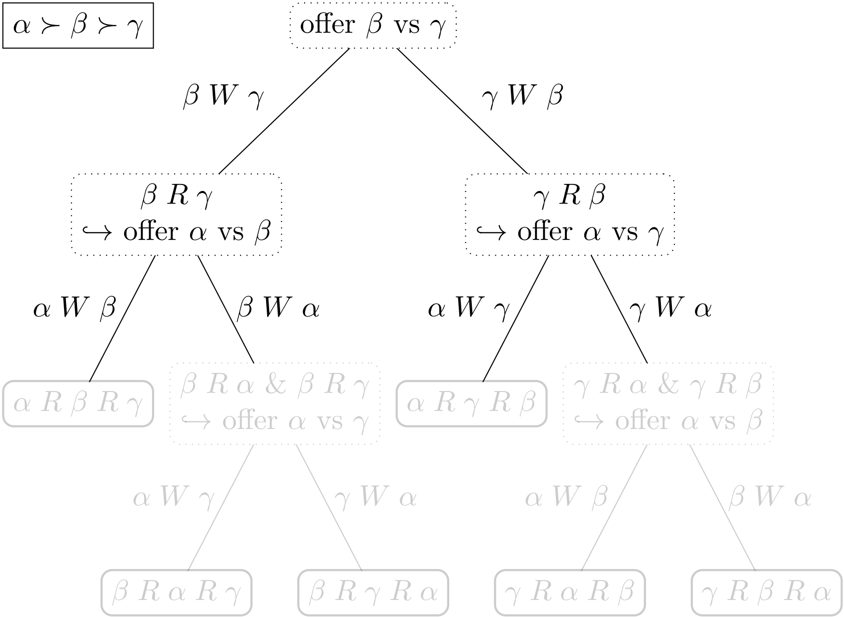
The converse is false:
a W-unimprovable ranking need not be W-efficient.

(counter-example: slide 38)

But only efficiency ensures unimprovability robustly across Ws:

Theorem 2.
A strategy is regret-free iff it is efficient.

(tightness: slide 40)

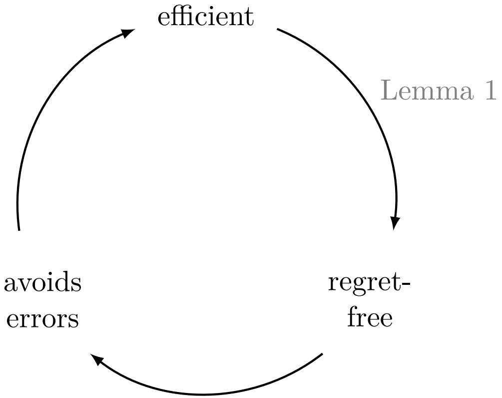

$\alpha \succ \beta \succ \gamma$
offer β vs γ
$\beta\, W\, \gamma$
$\gamma\, W\, \beta$
$\beta\, R\, \gamma$ $\hookrightarrow$ offer α vs β
$\gamma\, R\, \beta$ $\hookrightarrow$ offer α vs γ
$\alpha\, W\, \beta$
$\beta\, W\, \alpha$
$\alpha\, W\, \gamma$
$\gamma\, W\, \alpha$
$\alpha\, R\, \beta\, R\, \gamma$
$\beta\, R\, \alpha$ & $\beta\, R\, \gamma$ $\hookrightarrow$ offer α vs γ
$\alpha\, R\, \gamma\, R\, \beta$
$\gamma\, R\, \alpha$ & $\gamma\, R\, \beta$ $\hookrightarrow$ offer α vs β
$\alpha\, W\, \gamma$
$\gamma\, W\, \alpha$
$\alpha\, W\, \beta$
$\beta\, W\, \alpha$
$\beta\, R\, \alpha\, R\, \gamma$
$\beta\, R\, \gamma\, R\, \alpha$
$\gamma\, R\, \alpha\, R\, \beta$
$\gamma\, R\, \beta\, R\, \alpha$

Characterisation of behaviour

Theorem 3.
A strategy is regret-free iff
it never misses an opportunity or takes a risk.

(formal definitions: slide 41) (tightness: slide 42)

Proof of Theorems 2 & 3



(details: slide 43)

Odds and ends

What's special about insertion sort? (slide 44)

What if voters are strategic? (slide 37)

Alexander der fünfft

Thanks!

Gregorius der.xij.

Benedictus der.xiij.

(anti-)popes in 1409–10, from Schedel (1493)

‘More aligned with than’ for party lists

A more aligned party list
has a $\succ$-better candidate in k^{th} place, for each k.

$\implies$ the k^{th} $\succ$-best candidate who gets a seat is $\succ$-better at every realisation of uncertainty.

Converse if uncertainty also about who can take up a seat:

R is MAW $\succ$ than R' *if and only if*
the k^{th} $\succ$-best candidate who gets a seat is $\succ$-better
at every realisation of uncertainty.

(back to slide 4)

A characterisation of our protocol

A *ballot* is a set $B \subseteq \mathcal{X}$ of ≥ 2 alternatives.
An *election* is (B, V) where B is a ballot and $V : \{1, \ldots, I\} \to B$.
A *history* is a sequence of elections with distinct ballots.

Write $h \sqsubseteq h'$ iff h is a truncation of h'. For a set $\mathcal{H}$ of histories,
write $h \in \tau(\mathcal{H})$ ('h is terminal') iff $h \in \mathcal{H}$ and there is no $h' \sqsupset h$ in $\mathcal{H}$.

A *protocol* is a set $\mathcal{H}$ of ('permitted') histories s.t.

– $h \sqsubseteq h' \in \mathcal{H}$ implies $h \in \mathcal{H}$, and
– $((B_1, V_1), \ldots, (B_t, V_t)) \in \mathcal{H}$ implies $((B_1, V_1), \ldots, (B_t, V_t')) \in \mathcal{H} \;\; \forall V_t'$

and a map ρ that assigns a ranking to each terminal $h \in \mathcal{H}$.

$(\mathcal{H}, \rho)$ is a *restriction* of $(\mathcal{H}', \rho')$ iff $\tau(\mathcal{H}) \subseteq \tau(\mathcal{H}')$ and $\rho = \rho'|_{\tau(\mathcal{H})}$.

A characterisation of our protocol

For a history $h = ((B_t, V_t))_{t=1}^T$,

– write $x \; S^h \; y$ iff $x, y \in B_t$ and $|\{i : V_t(i) = x\}| \geq |\{i : V_t(i) = y\}| \;\; \exists t$

– say that h *gives the committee a say on* x, y iff $\{z_1, z_L\} = \{x, y\}$ for some sequence $z_1 \; S^h \; z_2 \; S^h \cdots S^h \; z_L$.

Proposition.

A protocol is a restriction of our transitive protocol iff it satisfies

(i) *binary ballots:* for any $((B_t, V_t))_{t=1}^T \in \mathcal{H}$, we have $|B_1| = \cdots = |B_T| = 2$.

(ii) *committee sovereignty:* at any terminal $h = ((B_t, V_t))_{t=1}^T \in \mathcal{H}$, if $|\{i : V_t(i) = x\}| > I/2$ and $y \in B_t$, then $x \; \rho(h) \; y$.

(iii) *democratic legitimacy:* every terminal $h \in \mathcal{H}$ gives the committee has a say on each pair of alternatives.

(back to slide 11)

Strategic voting

Each voter i has a preference $\succ_i$ over alternatives,
and prefers rankings more aligned with $\succ_i$.

A voter's *strategy* specifies how to vote at each history.
The *sincere strategy:* vote for your favourite. History-invariant!

Outcome of chair [voters] using σ $[\sigma_i, \sigma_{-i}]$ denoted $R(\sigma, \sigma_i, \sigma_{-i})$.

Definition.
A strategy σ_i is *dominant* iff for any alternative strategy σ_i',

($\nexists$) there exists no profile σ, σ_{-i} such that $R(\sigma, \sigma_i', \sigma_{-i})$ is distinct from, and MAW $\succ_i$ than, $R(\sigma, \sigma_i, \sigma_{-i})$.

($\exists$) there exists a profile σ, σ_{-i} such that $R(\sigma, \sigma_i, \sigma_{-i})$ is distinct from, and MAW $\succ_i$ than, $R(\sigma, \sigma_i', \sigma_{-i})$.

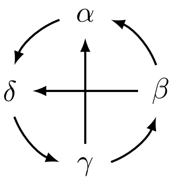
Proposition 4.

The sincere strategy is (uniquely) dominant.

(back to slide 13) (back to slide 32)

Counter-example to the converse of Lemma 1

$\mathcal{X} = \{\alpha, \beta, \gamma, \delta\}$ with $\alpha \succ \beta \succ \gamma \succ \delta$ and



The ranking $\alpha\ R\ \delta\ R\ \gamma\ R\ \beta \ldots$

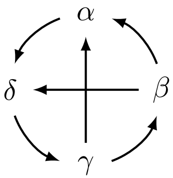
(– is W-feasible: offer $\{\alpha, \delta\}$, $\{\delta, \gamma\}$, $\{\gamma, \beta\}$.)

– is W-unimprovable,
since no other W-feasible ranking ranks α above β.
(Because there's only one directed path in W from α to β.)

– is not W-efficient, since $\delta\ R\ \beta$.

(back to slide 28)

Necessity of efficiency



and $\alpha \succ \beta \succ \gamma \succ \delta$.

non-W-efficient rankings feature sacrifices ($\delta \; R \; \beta$)

…which may pay off ($\alpha \; R \; \beta$) $\Longrightarrow$ W-unimprovable ranking

…or not $\Longrightarrow$ non-W-unimprovable ranking.

In fact, any sacrifice can fail to pay off
$\Longrightarrow$ inefficient strategies cannot be regret-free.

(back to slide 28)

Theorem 2 tightness

The characterisation in Theorem 2 is tight in the following sense:

Proposition 1.
For any majority will W and W-feasible W-efficient ranking R, some regret-free strategy has outcome R under W.

Thus for every majority will W,

$$\begin{aligned} & \{R : \exists \text{ regret-free strategy with outcome } R \text{ under } W\} \\ = & \{R : R \text{ is } W\text{-feasible and } W\text{-efficient}\} \end{aligned}$$

($\subseteq$ by Theorem 2, $\supseteq$ by Proposition 1)

(back to slide 28)

Formal definition of errors

Definition.
Let R be a non-total proto-ranking, and let $x \succ y$ be unranked.

(1) Offering $\{x, y\}$ for a vote *misses an opportunity (at R)* iff there is an alternative z s.t. $x \succ z \succ y$ and $y \not R z \not R x$.

(2) Offering $\{x, y\}$ for a vote *takes a risk (at R)* iff there is an alternative z s.t. either

- $z \succ y$, $x R z$ and $y \not R z$, or
- $x \succ z$, $z R y$ and $z \not R x$.

(back to slide 30)

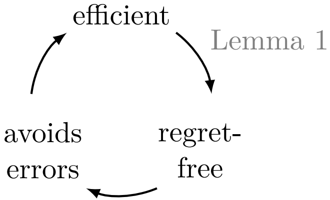
Theorem 3 tightness

Proposition 2.
After any error-free history,
there is a pair that can be offered without committing an error.

Yields tightness:
for any W and any sequence of pairs that is error-free under W,
some regret-free strategy offers this sequence under W.

(back to slide 30)

Proof of Theorems 2 & 3



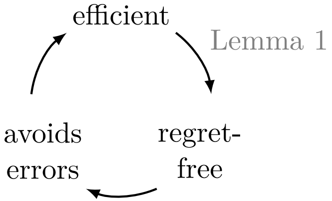
Avoids errors $\implies$ efficient: contra-positive.

– suppose σ not efficient $\implies$ for some W,
outcome R has $y\ R\ x$ despite $x \succ y$ and $x\ W\ y$.

– must be due to an unfavourable imposition of transitivity.

– argue that error-avoidance
precludes unfavourable impositions of transitivity.

Proof of Theorems 2 & 3



Regret-free $\implies$ no errors: contra-positive.

– suppose σ erroneously offers $x \succ y$ under some W

$$\implies \exists\, W' \quad \text{s.t.} \quad \begin{cases} y\ R\ x & \text{for outcome } R \text{ of } \sigma \text{ under } W' \\ x\ R'\ y & \text{for some other } W'\text{-feasible } R'. \end{cases}$$

– carefully construct W' and R' so that

every other pair z, w ranked 'right' by R
also ranked 'right' by R'.

(back to slide 31)

What's special about insertion sort?

For an alternative x, strategy σ and majority will W, write $R^\sigma(W)$ for the outcome of σ under W, and

$$N_x^\sigma(W) \coloneqq |\{y \in \mathcal{X} : x \succ y \text{ and } x \; R^\sigma(W) \; y\}|.$$

Definition.
Given $x \in \mathcal{X}$, σ is *better for x than* σ' iff

$$|\{W : N_x^\sigma(W) \geq k\}| \geq |\{W : N_x^{\sigma'}(W) \geq k\}| \quad \forall k \in \{1, \ldots, n-1\}.$$

If $\sigma \in \Sigma$ is better for x than each $\in \Sigma$, it is *best for x among Σ.*

Label the alternatives $\mathcal{X} \equiv \{1, \ldots, n\}$ so that $1 \succ \cdots \succ n$.

Theorem 4.
A strategy is outcome-equivalent to insertion sort iff
among all strategies, it is best for 1;
among such strategies, it is best for 2; and so on.

Intuition

Insertion sort is characterised by two properties:

- ‘bottom-up’: totally rank $\{k+1, \ldots, n\}$ before offering k
- ‘insertion’: votes involving k offered in a particular order

(Given ‘bottom-up’, ‘insertion’ is n&s for regret-freeness.)

‘Bottom-up’…

- maximises favourable impositions of transitivity involving 1;
- subject to that, maximises favourable … involving 2;
- and so on.

(back to slide 32)

The (recursive) amendment procedure

Amendment procedure: pit $n-1$ against n, then pit the winner against $n-2$, then pit the winner against $n-3$, and so on. Call the winner of the final round the *final winner*.

Recursive amendment procedure (a.k.a. 'selection sort'):

– run the AP on $\{1, \ldots, n\}$; call the final winner y_1.

– run the AP on $\{1, \ldots, n\} \setminus \{y_1\}$; call the final winner y_2.

– ...

The resulting ranking is $y_1 \; R \; y_2 \; R \cdots R \; y_{n-1} \; R \; y_n$.

Proposition 3.

Recursive amendment and insertion sort are outcome-equivalent.

(back to slide 32)

References I

Banks, J. S. (1985). Sophisticated voting outcomes and agenda control. *Social Choice and Welfare*, *1*(4), 295–306. doi:10.1007/BF00649265

Black, D. (1958). *The theory of committees and elections*. Cambridge: Cambridge University Press.

Copeland, A. (1951). *A reasonable social welfare function*. Notes from University of Michigan seminar on applications of mathematics to the social sciences.

Daniels, H. E. (1969). Round-robin tournament scores. *Biometrika*, *56*(2), 295–299. doi:10.2307/2334422

Farquharson, R. (1969). *Theory of voting*. New Haven, CT: Yale University Press.

Gershkov, A., Kleiner, A., Moldovanu, B., & Shi, X. (2019). *The art of compromising: Voting with interdependent values and the flag of the Weimar Republic*. Working paper, 9 Sep 2019.

References II

Gershkov, A., Moldovanu, B., & Shi, X. (2017). Optimal voting rules. *Review of Economic Studies*, *84*(2), 688–717. doi:10.1093/restud/rdw044

Gershkov, A., Moldovanu, B., & Shi, X. (2019). Voting on multiple issues: What to put on the ballot? *Theoretical Economics*, *14*(2), 555–596. doi:10.3982/TE3193

Gershkov, A., Moldovanu, B., & Shi, X. (2020). Monotonic norms and orthogonal issues in multidimensional voting. *Journal of Economic Theory*, *189*. doi:10.1016/j.jet.2020.105103

Kemeny, J. G. (1959). Mathematics without numbers. *Dædalus*, *88*(4), 577–591. JSTOR: 20026529

Kendall, M. G. (1955). Further contributions to the theory of paired comparisons. *Biometrics*, *11*(1), 43–62. doi:10.2307/3001479

References III

Kleiner, A., & Moldovanu, B. (2017). Content-based agendas and qualified majorities in sequential voting. *American Economic Review*, *107*(6), 1477–1506. doi:10.1257/aer.20160277

Miller, N. R. (1977). Graph-theoretical approaches to the theory of voting. *American Journal of Political Science*, *21*(4), 769–803. doi:10.2307/2110736

Moon, J. W., & Pullman, N. J. (1970). On generalized tournament matrices. *SIAM Review*, *12*(3), 384–399. doi:10.1137/1012081

Ordeshook, P. C., & Palfrey, T. R. (1988). Agendas, strategic voting, and signaling with incomplete information. *American Journal of Political Science*, *32*(2), 441–466. doi:10.2307/2111131

References IV

Rubinstein, A. (1980). Ranking the participants in a tournament. *SIAM Journal on Applied Mathematics*, *38*(1), 108–111. doi:10.1137/0138009

Schedel, H. (1493). *Register Des buchs der Croniken und geschichten mit figure und pildnussen von anbegin der welt bis auf dise unsere Zeit*. (M. Wolgemut & W. Pleydenwurff, Illustrators & G. Alt, Trans.). Nürnberg: Anton Koberger.

Slater, P. (1961). Inconsistencies in a schedule of paired comparisons. *Biometrika*, *48*(3–4), 303–312. doi:10.1093/biomet/48.3-4.303

Slutzki, G., & Volij, O. (2005). Ranking participants in generalized tournaments. *International Journal of Game Theory*, *33*(2), 255–270. doi:10.1007/s00182-005-0197-5

Wei, T.-H. (1952). *Algebraic foundations of ranking theory*. (doctoral thesis, University of Cambridge).

References V

Young, H. P. (1986). Optimal ranking and choice from pairwise comparisons. In B. Grofman & G. Owen (Eds.), *Information pooling and group decision making* (pp. 113–122). Greenwich, CT: JAI Press.

Young, H. P. (1988). Condorcet's theory of voting. *American Political Science Review*, *82*(4), 1231–1244. doi:10.2307/1961757

Young, H. P., & Levenglick, A. (1978). A consistent extension of Condorcet's election principle. *SIAM Journal on Applied Mathematics*, *35*(2), 285–300. doi:10.1137/0135023

Zermelo, E. (1929). Die Berechnung der Turnier-Ergebnisse als ein Maximumproblem der Wahrscheinlichkeitsrechnung. *Mathematische Zeitschrift*, *29*(1), 436–460. doi:10.1007/BF01180541